Cambridge International A Level

Cambridge International Examinations
Cambridge International Advanced Level

FURTHER MATHEMATICS **9231/02**

Paper 2 **For examination from 2017**

MARK SCHEME

Maximum Mark: 100

Specimen

This document consists of **16** printed pages.

CAMBRIDGE International Examinations

Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking *g* equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF	Any Equivalent Form (of answer is equally acceptable)
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
BOD	Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)
CAO	Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)
CWO	Correct Working Only – often written by a 'fortuitous' answer
ISW	Ignore Subsequent Working
MR	Misread
PA	Premature Approximation (resulting in basically correct work that is insufficiently accurate)
SOS	See Other Solution (the candidate makes a better attempt at the same question)
SR	Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1	A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through $\sqrt{}$" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.
PA –1	This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Question	Answer	Marks	Partial Marks	Guidance
1	Find 3 independent equations for T, R_A, R_B: Resolve horizontally: $R_B = T\cos\alpha$	2	**M1 A1**	
	Resolve vertically: $R_A = W + T\sin\alpha$	2	**M1 A1**	
	Take moments about A: $R_B\,3a\sin\theta = W(3a/2)\cos\theta$ $+ Ta(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$ *or* $+ Ta\sin(\alpha + \theta)$ *or* $+ T\,3a\cos\theta\sin\alpha$	2	**M1 A1**	(a may be omitted from moment eqns)
	Or: Take moments about B: $R_A 3a\cos\theta = W(3a/2)\cos\theta$ $+ T\,2a(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$ *or* $+ T\,2a\sin(\alpha + \theta)$ *or* $+ T\,3a\sin\theta\cos\alpha$	2	**(M1 A1)**	
	Or: Take moments about C: $R_A\,a\cos\theta + W(a/2)\cos\theta$ $= R_B\,2a\sin\theta$	2	**(M1 A1)**	
	Or: Take moments about D: $R_A\,3a\cos\theta - W(3a/2)\cos\theta$ $= R_B\,3a\sin\theta$	2	**(M1 A1)**	
	Solve for T, R_A, R_B (AEF in W and α): $T = W/2\sin\alpha$ *or* $\frac{1}{2}W\operatorname{cosec}\alpha$ $R_A = 3W/2$ $R_B = W/2\tan\alpha$ *or* $\frac{1}{2}W\cot\alpha$	3	**B1** **B1** **B1**	
		9		

Question	Answer	Marks	Partial Marks	Guidance
2	For A & B use conservation of momentum, e.g.: $2mv_A + mv_B = 2mu$	1	**M1**	(allow $2v_A + v_B = 2u$)
	Use Newton's law of restitution (consistent signs): $v_B - v_A = eu$	1	**M1**	
	Combine to find v_A and v_B : $v_A = (2 - e)\,u/3,\ \ v_B = 2(1 + e)\,u/3$	2	**A1 A1**	
		4		
	Find e from $v_A = \lvert v_B' \rvert$ with $v_B' = [-]\,0{\cdot}4\,v_B$: $(2 - e) = 0{\cdot}8\,(1 + e),\ \ e = 2/3$	**2**	**M1 A1**	
	EITHER: Equate times in terms of reqd. distance x: $(d - x)/\,v_A = d/v_B + x/v_B'$ (AEF) [$v_A = v_B' = 4u/9,\ \ v_B = 10u/9$]	2	**M1 A1**	speeds need not be found:
	Use $v_A = v_B' = 0{\cdot}4\,v_B$ to solve for x: $d - x = 0{\cdot}4\,d + x,\ \ x = 0{\cdot}3\,d$	2	**M1 A1**	
	OR: Find dist. moved by A when B reaches wall: $d_A = (d/v_B)\,v_A = 0{\cdot}4\,d$	(2)	**(M1 A1)**	
	Find reqd. distance x: $x = ½\,(d - d_A) = 0{\cdot}3\,d$	(2)	**(M1 A1)**	
		4		

Question	Answer	Marks	Partial Marks	Guidance
3	Find k by equating equilibrium tensions: $mg\,(a/2)/a = 2mg\,(3a/2 - ka)\,/\,ka$	2	**M1 A1**	(vertical motion can earn M1 only)
	$\frac{1}{2} = 3/k - 2$, $k = 6/5$ *or* $1{\cdot}2$	1	**A1**	
		3		
	Apply Newton's law at general point, e.g.: $m\,d^2x/dt^2 = -mg\,(a/2 + x)/a$ $+ 2mg\,(3a/2 - ka - x)\,/\,ka$ *or* $m\,d^2y/dt^2 = +mg\,(a/2 - y)/a$ $- 2mg\,(3a/2 - ka + y)\,/\,ka$	3	**M1 A2**	(lose A1 for each incorrect term)
	Simplify to give standard SHM eqn, e.g.: $d^2x/dt^2 = -(1 + 2/k)gx\,/\,a$ $= -8gx/3a$	1	**A1**	**S.R.**: B1 if no derivation (max 2/5)
	State or find period using $2\pi/\omega$ with $\omega = \sqrt{(8g/3a)}$: $T = 2\pi\sqrt{(3a/8g)}$ *or* $\pi\sqrt{(3a/2g)}$ *or* $3{\cdot}85\sqrt{(a/g)}$ *or* $1{\cdot}22\sqrt{a}$ [s]	1	**B1**√	(√ on ω)
		5		
	Substitute values in $v^2 = \omega^2(x_0^2 - x^2)$: $0{\cdot}7^2 = (8g/3a)\{(0{\cdot}2a)^2 - (0{\cdot}05a)^2\}$	2	**M1 A1**	
	Solve to find numerical value of a: $0{\cdot}49 = (8g/3) \times 0{\cdot}0375a$, $a = 0{\cdot}49$	1	**A1**	
		3		

Question	Answer	Marks	Partial Marks	Guidance
4	*EITHER:* Find tension at top from $F = ma$ vertically: $T = mu^2/a - mg$	1	**B1**	
	OR: Use energy at e.g. θ to upward vertical: $\frac{1}{2} mv^2 = \frac{1}{2} mu^2 + mga(1 - \cos\theta)$ Find tension T by using $F = ma$ radially: $T' = mv^2/a - mg\cos\theta$ Eliminate v^2: $= mu^2/a + mg(2 - 3\cos\theta)$ Find T at top by taking $\theta = 0$: $T = mu^2/a - mg$	1	**(B1)**	
	Find u_{min} by requiring $T \geqslant 0$ at top [or $T > 0$]: $u^2/a - g \geqslant 0$ so $u_{min} = \sqrt{ag}$	1	**B1**	A.G.
		2		
	Find v at bottom from conservation of energy: $\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mg \times 2a$ $v^2 = ag + 4ag,\ v = \sqrt{(5ag)}$	2	**M1** **A1**	
	Find new speed V from conservation of momentum: $m'V = mv$ with $m' = m + \frac{1}{4}m$ $V = 4v/5 = 4\sqrt{(ag/5)}$ *or* $(4/5)\sqrt{(5ag)}$ AEF	2	**M1** **A1**	
		4		

Question	Answer	Marks	Partial Marks	Guidance
	Find w^2 at angle θ from conservation of energy: $\frac{1}{2} m'w^2 = \frac{1}{2} m'V^2$ $- m'ga (1 + \cos\theta)$ $[w^2 = ag (6/5 - 2\cos\theta)]$	2	**M1 A1**	(condone m instead of m' here since cancels out)
	S.R. Invalid energy method (max 2/5): [gives $T' = (5mg/4)(2 - 3\cos\theta)$] $\frac{1}{2} m'w^2 = \frac{1}{2} mu^2$ $+ mga (1 - \cos\theta)$ $- \frac{1}{4}mga (1 + \cos\theta)$	1	**(B1)**	
	Find tension T' there by using $F = ma$ radially: $T' = m'w^2/a - m'g\cos\theta$	1	**B1**	
	Eliminate w^2: $= m'V^2/a - m'g (2 + 3\cos\theta)$	1	**A1**	
	Substitute for m' and V: $= (5mg/4)(6/5 - 3\cos\theta)$ *or* $3mg/2 - (15/4) mg\cos\theta$	1	**A1**	AEF
		5		
	Find $\cos\theta$ when string becomes slack from $T' = 0$: $\cos\theta = \frac{1}{3} \times 6/5 = 2/5$ *or* $0{\cdot}4$	**2**	**M1 A1**	**S.R.** Allow if found from $T' = mg (6/5 - 3\cos\theta)$

Question	Answer	Marks	Partial Marks	Guidance
5	Find or use sample mean <u>and</u> estimate population variance: $x = 222{\cdot}8 / 10 = 22{\cdot}28$ $s^2 = 4{\cdot}12 / 9$ $= 0{\cdot}458$ *or* $103/225$ *or* $0{\cdot}677^2$	1	**M1**	(allow biased here: $0{\cdot}412$ *or* $0{\cdot}642^2$)
	Find confidence interval e.g.: $22{\cdot}28 \pm t\sqrt{(0{\cdot}458 / 10)}$	2	**M1 A1**	(allow z in place of t)
	Use of correct tabular value: $t_{9,\,0.975} = 2{\cdot}26[2]$	1	**A1**	
	Evaluate C.I. correct to 3 s.f.: $22{\cdot}3 \pm 0{\cdot}48[4]$ *or* $[21{\cdot}8, 22{\cdot}8]$	1	**A1**	
		5		
6(i)	Find prob. p of head from mean = 2 × variance: $1/p = 2 \times (1 - p)/p^2, \quad p = ⅔$	**2**	**M1 A1**	**A.G.**
6(ii)	Find P(X = 4) (denoting $1 - p$ by q [= ⅓]): $P(X = 4) = q^3 \times p$ $= 2/81$ *or* $0{\cdot}0247$	**1**	**B1**	
6(iii)	Find or state P(X > 4): $P(X > 4)$ $[= 1 - (1 + q + q^2 + q^3) \times p$ $= 1 - (1 - q^4)\,] = q^4$ $= 1/81$ *or* $0{\cdot}0123$	**2**	**M1 A1**	

Question	Answer	Marks	Partial Marks	Guidance
6(iv)	Formulate condition for N: $1 - q^N > 0{\cdot}999$, $[\ (⅓)^N < 0{\cdot}001\ ]$	1	**M1**	
	Take logs (any base) to give bound for N: $N >\ \log 0{\cdot}001 / \log ⅓$	1	**M1**	
	Find $N_{\min}$: $N > 6{\cdot}29$, $N_{\min} = 7$	1	**A1**	($N < 6{\cdot}29$ *or* $N = 6{\cdot}29$ earns M2 A0)
		3		
7(i)	Find F(x) for $1 \leqslant x \leqslant 4$: $F(x) = (x^3 - 1)/63$	1	**B1**	
	Find G(y) from $Y = X^2$ for $1 \leqslant x \leqslant 4$: $G(y) = P(Y < y) = P(X^2 < y)$ $= P(X < y^{1/2}) = F(y^{1/2})$ $= (y^{3/2} - 1)/63$	2	**M1 A1**	(result may be stated)
	Find g(y) for corresponding range of y: $g(y) = y^{1/2}/42$	1	**A1**	**A.G.**
	Find or state corresponding range of y : $1 \leqslant y \leqslant 16$	1	**B1**	**A.G.**
		5		
7(ii)	Find median value m of Y: $(m^{3/2} - 1)/63 = ½$ $m = 32{\cdot}5^{\,2/3} = 10{\cdot}2$	**2**	**M1 A1**	
7(iii)	Find E(Y) [or equivalently E(X^2)]: $E(Y) = \int y\, g(y)\, dy = \int y^{3/2}\, dy /42$ $= [y^{5/2}]_1^{16} /105 = 1023/105$ $= 341/35$ *or* $9{\cdot}74$	**2**	**M1 A1**	

Question	Answer	Marks	Partial Marks	Guidance
8	Find mean of sample data [for use in Poisson distn.]: $\lambda = 220/100 = 2{\cdot}2$	1	**B1**	
	State (at least) null hypothesis (AEF): H_0: Poisson distn. fits data *or* $\lambda = 2{\cdot}2$	1	**B1**	
	Find expected values $100\lambda^r e^{-\lambda}/r!$ (to 1 d.p.): 11·080 24·377 26·814 19·664 10·8151 4·759 2·491	2	**M1 A1**	(ignore incorrect final value here for M1)
	Combine last two cells so that exp. value ⩾ 5: O_i: 3 E_i: 7·25	1	**M1***	
	Calculate value of χ^2 (to 2 d.p.; A1 dep M1*): $\chi^2 = 0{\cdot}076 + 2{\cdot}879 + 0{\cdot}653 + 1{\cdot}448$ $+ 0{\cdot}441 + 2{\cdot}491$ $= 7{\cdot}99$	2	**M1 A1**	(allow 7·95 if 1 d.p. exp.values used)
	State or use consistent tabular value (to 3 s.f.): 5 cells: $\chi_{3,0.95}{}^2 = 7{\cdot}815$ 6 cells: $\chi_{4,\,0.95}{}^2 = 9{\cdot}488$ (correct) 7 cells: $\chi_{5,\,0.95}{}^2 = 11{\cdot}07$	1	**B1**	
	State or imply valid method for conclusion e.g.: Accept H_0 if $\chi^2 <$ tabular value	1	**M1**	
	Conclusion (AEF, requires both values correct): Distn fits *or* $\lambda = 2{\cdot}2$	1	**DA1**	Not combining cells [so $\chi^2 = 8{\cdot}64$] can earn B1 B1 M1 A1 M0 M1 B1 M1 (max 7)
		10		

Question	Answer	Marks	Partial Marks	Guidance
9	Calculate gradient b_1 in $y - \bar{y} = b_1(x - \bar{x})$: $S_{xy} = 24\,879 - 472 \times 400/8$ $= 1\,279$ $S_{xx} = 29\,950 - 472^2/8 = 2\,102$ $b_1 = S_{xy} / S_{xx} = 0{\cdot}6085$ (3 s.f.)	2	**M1 A1**	
	Find regression line of y on x: $y = 400/8 + b_1 (x - 472/8)$ $= 50 + 0{\cdot}6085 (x - 59)$ $= 0{\cdot}6085x + 14{\cdot}1$	2	**M1 A1**	
	Find y when $x = 72$: $= 57{\cdot}9$ *or* 58 Allow use of regression line of x on y (since neither variable clearly independent): $S_{yy} = 21\,226 - 400^2/8 = 1\,226$ $b_2 = S_{xy} / S_{yy} = 1{\cdot}043$	2	**(M1 A1)**	
	$x = 472/8 + b_2 (y - 400/8)$	2	**(M1 A1)**	
	$= 1{\cdot}043\, y + 6{\cdot}85$ $Y = 62{\cdot}5$ *or* 62	1	**A1**	
		5		
	Find product moment correlation coefficient r: $r = 1\,279 / \sqrt{(2\,102 \times 1\,226)}$ *or* $\sqrt{(0{\cdot}6085 \times 1{\cdot}043)} = 0{\cdot}797$	2	**M1 A1***	
	State both hypotheses (B0 for r …): H_0: $\rho = 0$, H_1: $\rho \neq 0$	1	**B1**	
	State or use correct tabular two-tail r-value: $r_{8,\,5\%} = 0{\cdot}707$	1	**B1***	
	State or imply valid method for conclusion e.g.: Reject H_0 if $\lvert r \rvert >$ tab. value (AEF)	1	**M1**	

Question	Answer	Marks	Partial Marks	Guidance
	Correct conclusion : There is non-zero correlation	1	**DA1**	(AEF, dep A1*, B1*)
		6		

Question	Answer	Marks	Partial Marks	Guidance
10E	Find MI of lamina about Q: $I_{lamina} = \frac{1}{3}m\{(3a)^2 + (3a/2)^2\} + m(9a/2)^2$	2	**M1 A1**	[= (15/4 + 81/4) ma^2 = 24 ma^2]
	State or find MI of rod about Q: $I_{rod} = (\frac{1}{3} + 1)\, M\,(3a/2)^2$ [$= 3Ma^2$]	1	**B1**	
	Sum to find MI of object about Q: $I_1 = 24\,ma^2 + 3\,Ma^2$ $= 3\,(8m + M)\,a^2$	1	**A1**	**A.G.**
	Find MI of object about mid-point of PQ: $I_2 = (15/4 + 3^2)\,ma^2 + \frac{1}{3}\,M\,(3a/2)^2$ $= (51/4)\,ma^2 + \frac{3}{4}\,Ma^2$ $= \frac{3}{4}\,(17m + M)\,a^2$	2	**M1 A1**	**A.G.**
	Use eqn of circular motion to find $d^2\theta/dt^2$ for axis l_1: $[-]I_1\, d^2\theta/dt^2 = mg \times (9a/2) \sin\theta + Mg \times (3a/2) \sin\theta$ $[= (9m/2 + 3M/2)\, ga \sin\theta]$	2	**M1 A1**	
	[Approximate $\sin\theta$ by θ and] find ω_1^2 in SHM eqn: $\omega_1^2 = (3m + M)g \,/\, 2(8m + M)\,a$	1	**M1**	
	Find period T_1 for axis l_1 from $2\pi/\omega_1$: $T_1 = 2\pi\sqrt{\{2(8m + M)\,a \,/\, (3m + M)g\}}$	1	**A1**	(AEF)
	Use eqn of circular motion to find $d^2\theta/dt^2$ for axis l_2: $[-]I_2\, d^2\theta/dt^2 = mg \times 3a \sin\theta$	1	**M1**	
	[Approximate $\sin\theta$ by θ and] find ω_2^2 in SHM eqn: $\omega_2^2 = 4mg \,/\, (17m + M)\,a$	1	**M1**	
	Find period T_2 for axis l_2 from $2\pi/\omega_2$: $T_2 = 2\pi\sqrt{\{(17m + M)\,a \,/\, 4mg\}}$	1	**A1**	(AEF)
	Verify that $T_1 = T_2$ when $m = M$: (AEF) $T_1 = 2\pi\sqrt{(18\,a \,/\, 4g)} = T_2$	1	**B1**	[Taking $m = M$ throughout 2nd part can earn: M1 A1 M1 A0 M1 M1 A0 B1 (max 6/8)]
		8		

Question	Answer	Marks	Partial Marks	Guidance
10O	State hypotheses (B0 for $\overline{x}$ …), e.g.: H_0: $\mu_X = \mu_Y$, H_1: $\mu_X \neq \mu_Y$	1	**B1**	
	State assumption : Distributions have equal variances	1	**B1**	(AEF)
	Find sample mean <u>and</u> estimate popln variances: $x = 4{\cdot}2, y = 4.8$ $s_X^2 = (180 - 42^2/10) / 9$ $= 0{\cdot}4$ *or* $0{\cdot}6325^2$ $s_Y^2 = (281{\cdot}5 - 57{\cdot}6^2/12) / 11$ $= 0{\cdot}4564$ *or* $251/550$ *or* $0{\cdot}6755^2$	1	**M1**	(allow biased here: $0{\cdot}36$ *or* $0{\cdot}6^2$) (allow biased here: $0{\cdot}4183$ *or* $0{\cdot}6468^2$)
	Estimate (pooled) common variance: $s^2 = (9\, s_X^2 + 11\, s_Y^2) / 20$ *or* $(180 - 42^2/10 + 281{\cdot}5 - 57{\cdot}6^2/12) / 20$ $= 0{\cdot}431$ *or* $0{\cdot}6565^2$	2	**M1 A1**	(AEF) (note s_X^2 and s_Y^2 not needed explicitly)
	Calculate value of t (to 3 s.f.): [-] $t = (\overline{y} - \overline{x}) / s \sqrt{(1/10 + 1/12)}$ $= 2{\cdot}13$	2	**M1 A1**	
	State or use correct tabular t value: $t_{20,\, 0.975} = 2{\cdot}086$ [allow 2·09]	1	**B1***	(or can compare $\overline{y} - \overline{x} = 0{\cdot}6$ with $0{\cdot}586$)
	Correct conclusion: $t > 2{\cdot}09$ so mean masses not same	1	**DB1√**	(AEF, √ on t, dep *B1)
	S.R. Implicitly taking s_X^2, s_Y^2 as popln. variances: $z = (\overline{y} - \overline{x}) / \sqrt{(s_X^2/10 + s_Y^2/12)}$ $= 0{\cdot}6 / \sqrt{(0{\cdot}078} = 2{\cdot}15$	1	**(B1)**	(may also earn first B1 B1 M1)
		9		

Question	Answer	Marks	Partial Marks	Guidance
	Comparison with $z_{0.975}$ and conclusion: $2{\cdot}15 > 1{\cdot}96$ so mean masses not same	1	**(B1)**	(can earn at most 5/9)
	State hypotheses (B0 for $\bar{x}$ …), e.g.: H_0: $\mu_X = 3{\cdot}8$, H_1: $\mu_X > 3{\cdot}8$ *or* H_0: $\mu_X = \mu_Z$, H_1: $\mu_X > \mu_Z$	1	**B1**	
	Calculate value of t using s_X from above: $t = (4{\cdot}2 - 3{\cdot}8) / (s_X/\sqrt{10}) = 2{\cdot}0$	2	**M1 A1**	
	State or use correct tabular t value: $t_{9,\,0.95} = 1{\cdot}833$ [allow 1·83]	1	**B1***	(or can compare 0·4 with 0·367)
	Correct conclusion: $t > 1{\cdot}833$, so claim is justified *or* mean mass of Royals > mean mass of Crowns	1	**DB1**√	(A.E.F., √ on t, dep *B1)
		5		