FURTHER MATHEMATICS

Paper 9231/11
Paper 11

Key messages

- Candidates should show all the steps in their solutions, particularly when proving a given result.
- Candidates should make use of results derived or given in earlier parts of a question.
- They should be able to recall and apply skills from the 9709 Syllabus where appropriate.

General comments

Most candidates presented their work clearly and logically, realising the need for unambiguous mathematical communication. Scripts from the strongest candidates showed knowledge across all topics. Gaps in knowledge were evident in some scripts.

Comments on specific questions

Question 1

(i) This was very well done and good knowledge of implicit differentiation was seen.

(ii) Many candidates were accurate in evaluating the second derivative, though a few made errors with signs.

Question 2

(i) Most candidates successfully applied the formula for $\cos P + \cos Q$ to the denominator. Better responses also applied the formula for $\cos P - \cos Q$ to fully justify the given identity.

(ii) This part was done to a high standard with candidates writing out enough terms to justify cancellation, and most remembering to give their answer in terms of N not n.

(iii) Better responses gave a valid reason for divergence by stating that $\cos N$ oscillates.

Question 3

This question was well answered with the majority of candidates successfully finding $\overrightarrow{PQ}$ in terms of two parameters and then forming simultaneous equations. Some assumed that $\overrightarrow{PQ} = \begin{pmatrix} 1 \\ -1 \\ -6 \end{pmatrix}$ without justification.

Question 4

(i) Many candidates correctly used the idea of reverse differentiation. Reponses which involved the approach of integration by parts were less successful.

(ii) Almost all candidates applied integration by parts to I_n. Those who used the result from the previous part, or integrated x^n, successfully derived the given reduction formula. Others who tried to integrate e^{x^3} were less successful.

(iii) This part was well done with the majority of candidates accurately applying the reduction formula.

Question 5

(i) Most candidates accurately recalled the formula for surface area. Better responses fully simplified $\dot{x}$ and $\dot{y}$ before substituting them into the formula, which caused fewer errors and enabled a clear path to the given answer.

(ii) This part was well done with the majority of candidates accurately applying the given substitution and using the correct limits.

Question 6

(i) Almost all substituted for *x* into the original equation and cubed *y* + 1 to verify the result. The given equation in *y* was then used to correctly write down the value of S_3.

(ii) Most candidates used $\dfrac{\alpha^3\beta^3+\beta^3\gamma^3+\alpha^3\gamma^3}{\alpha^3\beta^3\gamma^3}$, with a few candidates opting to find a third cubic with reciprocal roots.

(iii) Several methods were employed in the final part, with a few candidates successfully working with the original equation in terms of *x*. Most candidates used the formula for the sum of squares and the given cubic in terms of *y* as intended. Some candidates who tried to recall complicated sigma formula made errors.

Question 7

Almost all candidates knew how to approach this question and completed it to a high standard. There was some inaccuracy when solving linear equations to find the constants and an occasional lack of clarity with notation. A few candidates used *x* as both a dependent and independent variable, and some gave expressions instead of equations as their answer.

Question 8

(i) Although most candidates clearly knew the requirement for an induction proof, sometimes details were omitted. Better responses checked the base case ($n = 1$ not $n = 2$), stated the assumption that the result is true for $n = k$ and proved the inductive step from the left hand side to the right hand side. They then summarised by stating that H_k implied H_{k+1}.

(ii) Almost all candidates knew that de Moivre's theorem related the series to the geometric progression in part **(i)**. However**,** it was common to see extra variables, such *m* or *n*, in the sum to infinity of the geometric progression which overcomplicated the expression and hindered progress. The strongest responses found the imaginary part of $\dfrac{-1}{2-1}$ or $\dfrac{z}{1-z}$ which led to the given answer.

Question 9

(i) Most candidates constructed the proof well although a few tried to square both sides or reversed the multiplication of **A** and **e**.

(ii) This part of the question was also well done, though a few candidates accepted zero eigenvectors without checking for errors in their working. Some candidates worked from **A**, others found **B** first, though the latter method produced more errors.

Question 10

(i) to (iv) Most candidates showed good knowledge of the algebraic and calculus skills involved, gaining all the marks available in the first four parts.

(v) Candidates found this part of the question more challenging. The better responses included sketches of the asymptotes meeting at a common point and positioned the branches correctly.

Question 11 – EITHER

(i) Some of the candidates who tackled this option found this part challenging, not appreciating the need to use $x = r\cos\theta$ for the distance from the line $\theta = \frac{1}{2}\pi$, and set $\frac{dx}{d\theta} = 0$ to find the maximum such distance. Most substituted appropriate values into $2\theta\tan\theta - 1$ and recorded a change in sign.

(ii) This part was very well done by the majority of candidates.

(iii) Most candidates sketched the correct shape of C_1 with the correct domain. Better responses had r strictly increasing and the correct intersection point with C_2.

(iv) Better responses correctly subtracted the area bounded C_2 and $\theta = \frac{1}{4}\pi$ from the area bounded C_1 and $\theta = \frac{1}{4}\pi$. Those who formed the wrong integral by having C_1 and C_2 interchanged in this subtraction often still gained credit for finding $\int \theta \sec^2\theta \, d\theta$ correctly using integration by parts.

Question 11 – OR

This was the more popular choice.

(i) **(a)** The majority of candidates were able to use row operations accurately, and better responses justified linear independence.

(b) Most candidates set $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ as a linear combination of the basis vectors and were able to show the given relationship directly. Some successfully used row operations. A few incorrectly assumed that $\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$ was in the null space.

(ii) It was pleasing to see how many candidates could solve this equation by separating the problem into two parts. Finding the particular solution and then identifying the basis for the null space was the most common method. A number of candidates combined the two parts and set up and solved equations or used the augmented matrix; working with equations did sometimes give rise to errors.

FURTHER MATHEMATICS

Paper 9231/12
Paper 12

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This was the more popular choice.

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(ii) It was pleasing to see how many candidates could solve this equation by separating the problem into two parts. Finding the particular solution and then identifying the basis for the null space was the most common method. A number of candidates combined the two parts and set up and solved equations or used the augmented matrix; working with equations did sometimes give rise to errors.

FURTHER MATHEMATICS

Paper 9231/13
Paper 13

Key messages

- When sketching graphs candidates should take care to show the sections of the graph that match the given domain. Sketches should show the correct behaviour at asymptotes and, for polar curves, at the pole.
- Candidates should make use of results derived or given in earlier parts of a question.
- They should be able to recall and apply skills from the 9709 syllabus where appropriate.

General comments

The majority of candidates demonstrated very good knowledge across the whole syllabus. They showed their working clearly and were accurate in their handling of algebra and calculus. They also showed understanding of linear algebra. It seemed that all were able to complete the paper in the time allowed, and the two options for **Question 11** were equally popular.

Comments on specific questions

Question 1

Most candidates showed a good knowledge of the structure of an induction proof, though some did not communicate all the steps clearly. Sometimes the proposition was assumed for *every* integer and several candidates wrote 3^k rather than 3^{3k}. Most manipulated the expressions well and better responses made the implication explicit in the final statement.

Question 2

(i) Most candidates drew the shape of the curve correctly and better responses showed the tangential nature of the graph at the pole.

(ii) This part of the question was completed to a very high standard by the majority of candidates. Clear working was shown by most. Some preferred to write the expression in terms of the original variable rather than change the limits.

Question 3

(i) It was common to see all five fifth roots of unity written correctly. A few candidates gave only the arguments.

(ii) Most candidates started the question successfully by solving a quadratic equation and finding the two values of z^5**.** Better responses found the fifth roots of both the values and were able to write down all ten answers. Some insightful solutions were seen using de Moivre's theorem to reduce the equation to $2\cos 5\theta = -1$.

Question 4

(i) Most candidates applied the method of differences correctly, showing enough working and terms to justify the given answer. A few candidates neglected to find the partial fractions before attempting to cancel terms.

(ii) Most candidates applied the result given in the previous part correctly. A few candidates chose to use the method of differences, again with mixed success. Some removed the N in the numerator or changed it to N^2.

Question 5

(i) This part was generally well done and good knowledge of linear algebra was demonstrated.

(ii) This part was also generally well done. A few candidates left their answer as a sum with two parameters, rather than writing down a basis.

Question 6

(i) With the exception of a small number of errors in division, the first part of the question was well done.

(ii) In finding the stationary points, most candidates differentiated correctly resulting in a quadratic expression which they set equal to zero. A few responses did not include the origin as a solution.

(iii) Better responses produced well drawn and labelled sketches, showing correct forms at infinity, and curves approaching asymptotes. Some sketches had misplaced branches, but most did have the general shape correct.

Question 7

(i) The majority of candidates found the common perpendicular and applied the formula for the shortest distance accurately. A few responses took the longer approach of finding points of intersection with the common perpendicular which produced more errors.

(ii) Most candidates knew how to find the angle required. They needed to adjust the answer they obtained, using the dot product correctly, to find the required angle.

Question 8

Almost all candidates knew how to approach this question and completed it to a high standard. There was some inaccuracy when solving linear equations to find the constants and some problems with notation. A few candidates used x as both a dependent and an independent variable, and some gave expressions instead of equations as their answer.

Question 9

(i) (ii) These parts were very well done by most candidates. They recalled the relationships between the roots and coefficients of the cubic equation and were able to use them accurately.

(iii) Those who used the cubic equation given in the question were usually successful, though a small number forgot to find S_0. Errors were seen when candidates did not recall the formula for $\sum\alpha^3$ correctly.

(iv) The most common approach was to work with the product of roots, realising that complex roots occur in conjugate pairs. Some candidates did not specify that b is the real root.

Question 10

(i) Most candidates differentiated $\cot^{n+1} x$ accurately using the chain rule. Some worked instead with $\sin x$ and $\cos x$, using the quotient rule, which led to the given result more directly. A few candidates lost track of the minus sign on the left hand side and this highlights the need for clear working when integrating both sides.

(ii) Most candidates recalled the formula the y-coordinate of the centroid accurately and applied the reduction formula given in part **(i)** to find I_2. A common error seen was $1-\frac{1}{4}\pi=\frac{3}{4}\pi$.

Question 11 – EITHER

(i) This part was done well and there were many accurate calculations when working with matrices. The usual method was to use diagonalisation but some impressive solutions using systems of equations were also seen.

(ii) There was less success in this part. Some responses unsuccessfully tried to use the answer to part **(i)**, whereas better responses used the correct eigenvalue and corresponding eigenvector as required.

(iii) Better responses derived the correct values by exploiting fully the properties of eigenvectors and eigenvalues. A few candidates did not exclude the negative solution to $b^2=1$.

Question 11 – OR

(i) (a) The majority of candidates could differentiate the equation implicitly and justify the given answer.

(i) (b) This was also completed to a very high standard with candidates going straight to the answer.

(iii) Most candidates realised the geometric nature of the series but some failed to recall the convergence condition correctly and $0<\ln a<1$ was sometimes seen.

(iv) The majority of candidates found the first derivative correctly using parametric differentiation. The attempts to find the second derivative varied greatly in length, with the better responses showing the required level of algebraic fluency and remembering to divide by $\frac{\mathrm{d}x}{\mathrm{d}t}$ after differentiating with respect to t.

FURTHER MATHEMATICS

Paper 9231/21
Paper 21

Key messages

To score full marks in this paper candidates must be well versed in both Mechanics and Statistics. Any preference between these two areas can only be exercised in the choice of the final optional question.

When a result is given in a question, candidates must take care to give sufficient detail in their working, so that the Examiner is in no doubt that the offered solution is clear and complete. In all questions, however, candidates are advised to show all their working, as credit is given for method as well as accuracy.

In Mechanics questions, a diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although a candidate is always free to draw their own diagram as well.

General comments

Almost all candidates attempted all the compulsory questions, and some very good answers were frequently seen. Most candidates opted for the Statistics option in **Question 11**.

Previous reports have stressed the need for candidates to set out their work clearly, and this advice has been heeded by most. This was particularly important in the unstructured **Question 4**.

The rubric for this paper specifies that non-exact numerical answers be given to 3 significant figures. Candidates would therefore be well-advised to work to a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures may result in an error in the third figure in the final answer. This is particularly the case in statistics problems where several different values are calculated, each depending on the previous one, as for example in **Question 9** and **Question 11**.

Comments on specific questions

Question 1

Candidates who knew the formulae for the radial and transverse components of acceleration usually scored full marks on this question. Many candidates either omitted the question or attempted to use formulae for constant acceleration.

Question 2

Since P is moving in simple harmonic motion, equating the ratio of the speeds at B and A to 2:1 using the standard result $v^2 = \omega^2\left(a^2 - x^2\right)$ leads to an equation for a^2 and hence the amplitude of the motion. Equating the maximum acceleration $\omega^2 a$ to the given value of 1 leads to the value of ω. Finally the speed at O is equal to ωa. Common errors in this part were to use the ratio 1:2 or to fail to square the 2 when using the formula for v^2.

The required time from A to B can be found in different ways. It is important to consider whether the appropriate expression for each of the distances being found is $a\sin\omega t$ or $a\cos\omega t$.

Question 3

Many candidates were able to formulate two simultaneous equations for the speeds of A and B after the first collision, by means of conservation of momentum and Newton's law of restitution. There were a few sign errors in the equations, and candidates are reminded that a diagram with masses and velocities, with magnitude and direction clearly marked, is invaluable in avoiding such errors. Having found these speeds, the process then needs to be repeated for the second collision, between B and C. In this case, since the speed of B after the collision is zero, the simultaneous equations lead to a quadratic equation in e, the coefficient of restitution.

The total kinetic energy lost by the three spheres as a result of the two collisions can be found in three ways: by finding the loss in kinetic energy of each sphere individually, or by finding the difference between the total initial kinetic energy and the total final kinetic energy, or by finding the kinetic energy lost in each collision.

Question 4

As in all questions of this type, candidates are well advised to first identify all the forces acting on the rod, preferably showing them on the given diagram, or on their own diagram. This will help the candidate to ensure that they include all the relevant forces when taking moments or resolving. Candidates were required to find the tension in the string and the value of the coefficient of friction, in either order, leaving them free to choose their own method. The easiest and most direct method of solution is first to take moments about the point A, thereby eliminating the forces at A and obtaining an equation involving only the tension T and the weight W. This enables T to be found in terms of W. The next step is to resolve the forces horizontally and vertically, leading to expressions for the frictional force and the reaction force at A and hence the value of the coefficient of friction.

Of course, it is possible to take moments about several other points and many candidates did indeed do so. Almost invariably, however, the candidates who did this were unsuccessful in isolating the forces that they required from the resulting moments equations.

Question 5

Almost all candidates were able to make an attempt at finding the moment of inertia of the object and did so by finding the moment of inertia for each of the three component parts: the rod, the ring and the hollow sphere. The last two of these required the use of the parallel axes theorem. Many candidates set out their work clearly, with each of the contributions to the total moment of inertia clearly identified. Such detail is important when the final result is given in the question. Candidates who simply write down a sum of terms run considerable risk, since an error in one term which still leads to the given correct answer does cast doubt on the validity of the whole process.

Question 6

This question on the geometric distribution produced good answers by many candidates. In the final part, the equation to be solved is $1-q^{N-1} < 0.95$. A common error was to have the power of q as N. In other cases, some incorrect manipulation of the inequality sign frequently led to $N > 8.39$ rather than $N < 8.39$ leading to an answer of 9 instead of 8.

Question 7

Many candidates were able to find the distribution function of X for $1 \leqslant x \leqslant 3$, but not all completed the description by including the value 0 for $x < 1$ and the value 1 for $x > 3$. The lower and upper quartiles were found accurately by many candidates, and then subtracted to give the interquartile range as equal to 1.

Question 8

Most candidates recognised this as an example of a paired-sample t-test. The method involves finding the difference between the two times for each of the runners and using these differences as the sample test values. The sample mean and an estimate for the population variance for these differences form the basis of a small sample t-test.

A minority of candidates did not recognise the information as a paired sample and used instead a two-sample t-test. This is not an appropriate approach for this type of data.

Question 9

This question tests the appropriateness of the given probability density function as a fit to the given data. The expected frequencies were given in the table, and candidates were asked to verify just one of them. Most candidates were able to do this, the only error being a tendency to round prematurely. As with any verification, all the necessary working must be shown to a suitable degree of accuracy, in this case at least 3 decimal places.

The final expected frequency in the table is less than 5 and this means that the last two columns must be combined, giving 12.65, before the chi-squared test value is calculated. A significant minority of candidates omitted to do this.

Question 10

The method for the first part of this question involves finding the values of S_{xy} and S_{yy} in terms of *q* and then equating S_{xy}/S_{yy} to the gradient $\frac{5}{4}$ of the given regression line of *x* on *y*. Many candidates approached the solution in this way and worked accurately to obtain the integer value 5 for *q*. Other candidates applied the correct method but made errors in the algebraic manipulation.

The final two parts of the question were attempted by many candidates, using either the correct value of *q* or the value that they had found in **part (i)**.

Question 11 (Mechanics)

This optional question was attempted by only a small minority of candidates, but these attempts were usually of a very high standard. To find the speeds of *P* and *Q* after their collision required three steps: conservation of energy for *P* from its starting point to its lowest point (giving the speed of *P* immediately before the collision), conservation of energy for *Q* from its starting point vertically below *O* (giving the velocity of *Q* immediately after the collision), and conservation of momentum at the collision (giving the speed of *P* immediately after the collision).

The second part of the question required the use of conservation of energy for *P* from immediately after the collision to the point where it loses contact with the inner surface of the sphere, and the use of Newton’s second law of motion at this point of lost contact.

Question 11 (Statistics)

This was by far the more popular choice of optional question and the solutions were of a high standard.

The given confidence interval is used to find an unbiased estimate for the population variance of cherries of Type *A*. Many candidates made good progress, but some made an incorrect choice for the tabular *t*-value. Others used a *z*-value, which is not appropriate for a sample size of only 8.

The significance test was usually carried out successfully and accurately. The instruction to assume that the population variances for the two types of cherry are equal implies that a pooled common variance estimate is appropriate. Only a very few candidates worked with the assumption of unequal population variances.

FURTHER MATHEMATICS

Paper 9231/22
Paper 22

Key messages

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A minority of candidates did not recognise the information as a paired sample and used instead a two-sample t-test. This is not an appropriate approach for this type of data.

Question 9

This question tests the appropriateness of the given probability density function as a fit to the given data. The expected frequencies were given in the table, and candidates were asked to verify just one of them. Most candidates were able to do this, the only error being a tendency to round prematurely. As with any verification, all the necessary working must be shown to a suitable degree of accuracy, in this case at least 3 decimal places.

The final expected frequency in the table is less than 5 and this means that the last two columns must be combined, giving 12.65, before the chi-squared test value is calculated. A significant minority of candidates omitted to do this.

Question 10

The method for the first part of this question involves finding the values of S_{xy} and S_{yy} in terms of q and then equating S_{xy}/S_{yy} to the gradient $\frac{5}{4}$ of the given regression line of x on y. Many candidates approached the solution in this way and worked accurately to obtain the integer value 5 for q. Other candidates applied the correct method but made errors in the algebraic manipulation.

The final two parts of the question were attempted by many candidates, using either the correct value of q or the value that they had found in **part (i)**.

Question 11 (Mechanics)

This optional question was attempted by only a small minority of candidates, but these attempts were usually of a very high standard. To find the speeds of *P* and *Q* after their collision required three steps: conservation of energy for *P* from its starting point to its lowest point (giving the speed of *P* immediately before the collision), conservation of energy for *Q* from its starting point vertically below *O* (giving the velocity of *Q* immediately after the collision), and conservation of momentum at the collision (giving the speed of *P* immediately after the collision).

The second part of the question required the use of conservation of energy for *P* from immediately after the collision to the point where it loses contact with the inner surface of the sphere, and the use of Newton's second law of motion at this point of lost contact.

Question 11 (Statistics)

This was by far the more popular choice of optional question and the solutions were of a high standard.

The given confidence interval is used to find an unbiased estimate for the population variance of cherries of Type *A*. Many candidates made good progress, but some made an incorrect choice for the tabular *t*-value. Others used a *z*-value, which is not appropriate for a sample size of only 8.

The significance test was usually carried out successfully and accurately. The instruction to assume that the population variances for the two types of cherry are equal implies that a pooled common variance estimate is appropriate. Only a very few candidates worked with the assumption of unequal population variances.

FURTHER MATHEMATICS

Paper 9231/23
Paper 23

Key messages

To score full marks in this paper candidates must be well versed in both Mechanics and Statistics. Any preference between these two areas can only be exercised in the choice of the final optional question.

When a result is given in a question, candidates must take care to give sufficient detail in their working, so that the Examiner is in no doubt that the offered solution is clear and complete. In all questions, however, candidates are advised to show all their working, as credit is given for method as well as accuracy.

In Mechanics questions, a diagram is often an invaluable tool in helping a candidate to make good progress. This is particularly the case when forces or velocities are involved. If a diagram is given on the question paper, then it may be sufficient to annotate that diagram, although a candidate is always free to draw their own diagram as well.

General comments

Almost all candidates attempted all the compulsory questions. Some very good responses were frequently seen. Most candidates opted for the Statistics option in **Question 11**.

Previous reports have stressed the need for candidates to set out their work clearly, and this advice has been heeded by most. The exceptions on this paper were in **Question 5** and **Question 10(i).**

Question 5 proved challenging for a significant number of candidates. Most added the direction and point of application of each force to the diagram on the question paper. However, there were then two common sources of error. Firstly, some candidates did not follow the instruction in the question to resolve along the rod. It should be noted that when a method is specified in a question, then marks cannot be earned by using a different method. Secondly, some candidates seemed to take moments about several points and to resolve in various directions, and so have a collection of equations involving several unknown forces, distances and angles. It is important to note that it is always worth thinking before embarking on this process, to determine a strategy. For example, taking moments about a point that eliminates several forces is usually a good starting point.

The rubric for this paper specifies that non-exact numerical answers be given to 3 significant figures. Candidates would therefore be well-advised to work to a greater degree of accuracy while working towards the final answer. Premature rounding or working to only 3 significant figures may result in an error in the third figure in the final answer. This is particularly the case in statistics problems where several different values are calculated, each depending on the previous one as, for example, in **Question 11**.

Comments on specific questions

Question 1

There are various approaches to finding the time T for which the bullet is in the barrier. The most straightforward is to use Newton's second law of motion to find the acceleration of the bullet and then find T from $v = u + at$. A second approach is to find the impulse from the change in momentum and then find T from impulse = Ft. The majority of candidates were able to find T correctly.

Question 2

Most candidates were able to formulate two equations relating the tension and speed, one at each of the points *A* and *B*, using Newton's second law of motion. This was followed by applying conservation of energy from *A* to *B*, giving an expression for the speed at *B*. These three equations are then combined together with the given information relating the two tensions to find the cosine of angle *α.* Many candidates convincingly arrived at the given result of ¾. The most common errors were sign errors in the initial three equations.

Finding the tension in the string when *P* is at *B* involved a simple substitution of the given result into one of the equations previously obtained.

Question 3

Many candidates were able to formulate two simultaneous equations for the speeds of *A* and *B* after the first collision, by means of conservation of momentum and Newton's law of restitution. There were a few sign errors in the equations, and candidates are reminded that a diagram with masses and velocities, with magnitude and direction clearly marked, is invaluable in avoiding such errors. Having found these speeds, the process then needs to be repeated for the second collision between *B* and *C*. There were again some sign errors, and also some algebraic errors, but many candidates obtained the correct final expressions for the speeds of all three spheres.

The speeds of *A* and *C* are then equated, resulting in a quadratic equation for the coefficient of restitution. This equation has two solutions, $\frac{1}{3}$ and – 3, the second of which must be rejected because the coefficient of restitution is a positive quantity.

Question 4

Almost all candidates were able to make an attempt at finding the moment of inertia of the object and did so by finding the moment of inertia for each of the three component parts: the rod, the large sphere and the small sphere. Each of these parts involves the use of a standard formula and the parallel axes theorem.

There were two common errors: firstly, the use of an incorrect formula, and secondly an incorrect distance used in the application of the parallel axes result. It was not always clear where candidates had found their standard formulae: it was not uncommon for them to be similar to known formulae but with halves and thirds included for unclear reasons. The final expression for the moment of inertia of the object was not given on this paper. This prompted many candidates to explain what they were doing in detail, which is to be encouraged more generally.

The second part of this question was challenging for some candidates, with a minority making no attempt at all. The best approach was to use conservation of energy. This involves finding the loss of potential energy of the object as it falls from its starting position, with the rod at 60° to the downward vertical, to the position where the rod is vertical and the angular speed is greatest. This potential energy loss may be calculated by considering the object as three separate entities or by first finding the centre of mass of the whole. The distance fallen involves a factor (1-cos 60°) and it was finding this distance that was the most problematic for candidates.

Question 5

As in all questions of this type, candidates are well advised to first identify all the forces acting on the rod, preferably showing them on the given diagram, or on their own diagram. This will help the candidate to ensure that they include all the relevant forces when taking moments or resolving. In the first part of the question, candidates were told the method that was to be used to find the normal component of the reaction of the bowl on the rod at *B*. The instruction 'by resolving parallel to the rod' was designed to help the candidate, by excluding the need to involve the reaction at the rim of the bowl. Those candidates who followed the instruction were usually successful in obtaining the given result. However, many candidates ignored the instruction and took moments about a variety of points and/or resolved horizontally and vertically. As the option to solve the problem 'otherwise' was not offered in the question, candidates should be aware that alternative methods may not receive full credit.

In the second part of the question, any method was acceptable, although the obvious way was to resolve perpendicular to the rod. Those candidates who took moments rarely made any progress, because the distances involved the unknown x.

Finding the distance x needed a moments equation. As always, candidates are well advised to consider an appropriate point about which to take moments, ideally involving as few forces as possible. In this case, B was a good choice, because it involved only W and the reaction force found in the first part of the question.

Question 6

Most candidates produced correct answers to this question. There was some confusion between the probability density function and the distribution function in the first part.

Question 7

This question on the geometric distribution produced good answers by many candidates. The probability of obtaining two heads when a pair of coins is thrown is $\frac{1}{4}$, but a minority of candidates thought that it was $\frac{1}{2}$. These candidates were still able to earn method marks.

In the last part, the equation to be solved is $1 - q^{N-1} > 0.95$. A common error was to have the power of q as N. In other cases, incorrect manipulation of the inequality sign frequently led to $N < 11.4$ rather than $N > 11.4$, giving an answer of 11 instead of 12.

Question 8

This question involved the application of the chi-squared test to a contingency table and many candidates performed well. Calculation of the chi-squared value of 3.38 was usually accurate, and most identified the appropriate tabular value as 4.605 (4.61). The test leads to the acceptance of the null hypothesis, that the type of holiday is independent of the salesman. Some candidates did not state the null hypothesis, and others talked about association. It is always worth noting the wording of the question, which in this case clearly asks for a test about independence. The conclusion should always involve a statement in the context of the question.

Question 9

As in all such tests, the hypotheses required should always be stated in terms of the population mean and not the sample mean. For this small sample, the data leads to a t-value of 1.99. This is a one-tail test, looking at a potential increase in yield. Comparison of the calculated t-value with 1.83 leads to acceptance of the alternative hypothesis, supporting the farmer's claim. This question produced very good answers by almost all candidates.

Question 10

Many candidates found the correct equation for the regression line of y on x. However, solutions to this question were not presented clearly and were often not easy to follow.

For the significance test, candidates stated the null and alternate hypotheses in terms of ρ, used the correct tabular value of 0.621 and reached the correct conclusion that there was insufficient evidence to support a positive correlation.

There are various statements that can be made to support the conclusion that the estimate in the final part is unreliable. The previous part has led to the conclusion that there is little evidence to support positive correlation, the value of r is small, the value of r is not close to one, 6.0 is not close to the mean of x. Any one of these is acceptable.

Question 11 (Mechanics)

This optional question was attempted by a minority of candidates. Almost all were able to equate the kinetic and elastic potential energies to derive the given expression for k in terms of a, g and u. To show that P performs simple harmonic motion requires an application of Newton's second law at a general point to find an expression for the second derivative of x with respect to t in terms of x. In this case, $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{kg}{a}x$, or $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{16u^2}{a^2}x$. From this, the value of ω and the period of the motion can be deduced.

There are several ways to find the time that elapses before P first loses 25 per cent of its initial kinetic energy. The most straightforward is to find the velocity at this point from the loss in energy. Then use the simple harmonic motion formulae, for example $x = \frac{1}{4}a \sin\omega t$, and the equivalent for velocity to find t. Those candidates who attempted this part usually answered well, but many candidates chose not to attempt this part of the question.

Question 11 (Statistics)

Most candidates were able to calculate an estimate for the combined variance of the two distributions and the test statistic z as 1.19. Some candidates assumed a common population variance and calculated a pooled estimate, leading to a z value of 1.09. In this latter case, a statement of the assumption was required.

The final step was to find the limiting value for α, based on a one-tail test. Some candidates were able to reverse the usual process and complete this successfully.