Cambridge International Examinations
Cambridge International Advanced Level

FURTHER MATHEMATICS **9231/13**

Paper 1 **May/June 2016**

MARK SCHEME

Maximum Mark: 100

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge will not enter into discussions about these mark schemes.

Cambridge is publishing the mark schemes for the May/June 2016 series for most Cambridge IGCSE®, Cambridge International A and AS Level components and some Cambridge O Level components.

Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking *g* equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a 'fortuitous' answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Qu	Solution	Marks
1	$\frac{1}{3}\left(\frac{1}{3r+1}-\frac{1}{3r+4}\right)=\frac{1}{3}\left(\frac{(3r+4)-(3r+1)}{(3r+1)(3r+4)}\right)=\frac{1}{(3r+1)(3r+4)}$ **AG**	**B1** [1]
	$S_N=\frac{1}{3}\left[\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{10}\right)+\ldots+\left(\left(\frac{1}{3N+1}\right)-\frac{1}{3N+4}\right)\right]=\frac{1}{3}\left(\frac{1}{4}-\frac{1}{3N+4}\right)$	**M1 A1**
	$\Rightarrow S=\frac{1}{12}$	**A1**
	$\Rightarrow S-S_N=\frac{1}{3(3N+4)}<\frac{1}{10000}$	**M1**
	$\Rightarrow 3N+4>\frac{10000}{3}\Rightarrow N>1109\ \frac{7}{9}$. Thus least N is 1110.	**A1** [5]
2	With $n=3$, $\frac{1}{2}n(n-3)=0$	**M1**
	A triangle has no diagonals ⇒ H_3 is true.	**A1**
	Assume H_k is true: A k-gon has $\frac{1}{2}k(k-3)$ diagonals for some integer ≥ 3	**B1**
	Adding an extra vertex, a further $(k-1)$ diagonals can be drawn.	**M1**
	$\frac{1}{2}k(k-3)+k-1=\frac{k^2-3k+2k-2}{2}=\frac{(k+1)(k-2)}{2}$	**A1**
	$=\frac{1}{2}(k+1)(k+1-3)$ (So $H_k\Rightarrow H_{k+1}$)	**A1** [6]
	⇒H_n is true for all integers $n\geqslant 3$.	
3	$\begin{vmatrix}k&1&1\\1&k&1\\1&1&k\end{vmatrix}=0\Rightarrow k^3-3k+2=0$	**M1A1**
	$(k-1)^2(k+2)=0\Rightarrow k=1,-2$	**M1A1** [4]
	$k=1:\Rightarrow x+y+z=2$ and $x+y+z=-1\Rightarrow$ inconsistent.	**B1**
	$k=-2:\Rightarrow x=z-1$	**M1**
	Hence $(x,y,z)=([-1+t],t,t)$	**A1** [3]

Qu	Solution	Marks
4	$\frac{dy}{dx}=\frac{2x}{1-x^2}$	**B1**
	$1+\left(\frac{dy}{dx}\right)^2=1+\frac{4x^2}{\left(1-x^2\right)^2}=\frac{1-2x^2+x^4+4x^2}{\left(1-x^2\right)^2}=\frac{\left(1+x^2\right)^2}{\left(1-x^2\right)^2}$ (**AG**)	**B1** [2]
	$\frac{1+x^2}{1-x^2}\equiv\frac{1}{1+x}+\frac{1}{1-x}-1$	**M1A1** [2]
	$s=2\int_0^{\frac{1}{2}}\left(\frac{1}{1+x}+\frac{1}{1-x}-1\right)dx$ (oe)	**M1**
	$=2\left[\ln\left(\frac{1+x}{1-x}\right)-x\right]_0^{\frac{1}{2}}$	**M1A1**
	$=2\ln 3-1$ (AEF)	**A1** [4]
5	$\frac{dy}{dx}=\frac{\left(x^2-9\right)-2x\left(x+2\right)}{\left(x^2-9\right)^2}$	**B1**
	$=\frac{-x^2-4x-9}{\left(x^2-9\right)^2}=\frac{-\left(x+2\right)^2-5}{\left(x^2-9\right)^2}\Rightarrow\frac{dy}{dx}<0$	**M1A1** [3]
	Asymptotes: $x=\pm3$; $y=0$	**B1B1** [2]
	Sketch: Axes and asymptotes ; Outside branches Middle branch, showing $\left(0,-\frac{2}{9}\right)$ and $\left(-2,0\right)$. (Deduct at most 1 mark for poor forms at infinity.)	**B1B1** **B1** [3]

Qu	Solution	Marks
6	$\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^n\left(4-x^2\right)^{\frac{3}{2}}\right\}=-3x^{n+1}\left(4-x^2\right)^{\frac{1}{2}}+nx^{n-1}\left(4-x^2\right)^{\frac{3}{2}}$	B1
	$-3x^{n+1}\left(4-x^2\right)^{\frac{1}{2}}+nx^{n-1}\left(4-x^2\right)^{\frac{1}{2}}\left(4-x^2\right)$	M1
	$\Rightarrow\left[x^n\left(4-x^2\right)^{\frac{3}{2}}\right]_0^2=0=-3I_{n+1}+4nI_{n-1}-nI_{n+1}$	M1
	$\Rightarrow(n+3)I_{n+1}=4nI_{n-1}$ **(AG)**	A1 [4]
	$I_1=\left[-\frac{1}{3}\left(4-x^2\right)^{\frac{3}{2}}\right]_0^2=\frac{8}{3}$	M1A1
	$5I_3=8I_1\Rightarrow I_3=\frac{64}{15}$	M1A1√ [4]
7	$\sqrt{x^2+y^2}=\frac{1}{1-\frac{x}{\sqrt{x^2+y^2}}}\Rightarrow\sqrt{x^2+y^2}-x=1$	M1A1
	$\Rightarrow y^2=1+2x$ or equivalent RHS	A1 [3]
	Sketch: Parabola symmetrical about *x*-axis. Intercepts at $\left(-\frac{1}{2},0\right)$ and $(0,\pm1)$.	B1 B1
	Shading correct area	B1 [3]
	Recognises $\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{1}{(1-\cos\theta)^2}d\theta$ as the area of the region between parabola and *y*-axis.	
	$=2\times\int_{-\frac{1}{2}}^{0}(1+2x)^{\frac{1}{2}}\,\mathrm{d}x$	M1
	$=2\left[\frac{1}{3}(1+2x)^{\frac{3}{2}}\right]_{-\frac{1}{2}}^{0}=\frac{2}{3}$	M1A1 [4]

Qu	Solution	Marks
8	Let S_n denote $\alpha^n + \beta^n + \gamma^n$. $S_2 = S_1^2 - 2\sum\alpha\beta = 1^2 - 2\times(-1) = 3$ $S_3 = S_2 + S_1 + 15 = 3 + 1 + 15 = 19$ (**AG**) Alt method: $\sum\alpha = 1$, $\Sigma\alpha\beta = -1$, and $\alpha\beta\gamma = 5$ **M1, A1** $S_3 = S_1^3 - 3\Sigma\alpha\,\Sigma\alpha\beta + 3\alpha\beta\gamma = 19$ **M1, A1**	**M1A1** **M1A1** [4]
	$S_4 = S_3 + S_2 + 5S_1 = 19 + 3 + 5 = 27$	**M1A1** [2]
	$x = \frac{z-1}{z} = 1 - \frac{1}{z} \Rightarrow \frac{1}{z} = 1 - x \Rightarrow z = \frac{1}{1-x}$ Hence $\frac{1}{(1-x)^3} - \frac{1}{(1-x)^2} - \frac{1}{(1-x)} - 5 = 0$ is the required equation. $\Rightarrow 1-(1-x)-(1-x)^2-5(1-x)^3 = 0 \Rightarrow \ldots \Rightarrow 5x^3 - 16x^2 + 18x - 6 = 0$	**B1** **M1** **M1A1** [4]
9	$\left(z+\frac{1}{z}\right)^4 = \left(z^4+\frac{1}{z^4}\right) + 4\left(z^2+\frac{1}{z^2}\right) + 6$ Taking $z = \cos\theta + \mathrm{i}\sin\theta$, $(2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$ $\Rightarrow \cos^4\theta = \frac{1}{16}(2\cos 4\theta + 8\cos 2\theta + 6) = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$ (**AG**)	**M1** **M1A1** **A1** [4]
	$\left(z-\frac{1}{z}\right)^4 = \left(z^4+\frac{1}{z^4}\right) - 4\left(z^2+\frac{1}{z^2}\right) + 6$ $\Rightarrow (2\mathrm{i}\sin\theta)^4 = 2\cos 4\theta - 8\cos 2\theta + 6$ $\Rightarrow \sin^4\theta = \frac{1}{16}(2\cos 4\theta - 8\cos 2\theta + 6) = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$	**M1** **M1A1** **A1** [4]
	$\int_0^{\frac{1}{8}\pi}(\cos^4\theta + \sin^4\theta)\,\mathrm{d}\theta = \frac{1}{4}\int_0^{\frac{1}{8}\pi}(\cos 4\theta + 3)\,\mathrm{d}\theta$ $= \frac{1}{4}\left[\frac{\sin 4\theta}{4} + 3\theta\right]_0^{\frac{1}{8}\pi} = \frac{1}{16} + \frac{3}{32}\pi$ (oe)	**M1** **M1A1** [3]

Qu	Solution	Marks
10	$x\frac{dy}{dx} = x\frac{dy}{du}\times\frac{du}{dx} = x\frac{dy}{du}\times\frac{1}{x} = \frac{dy}{du}$ **(AG)**	B1
	$x^2\frac{d^2y}{dx^2} = x^2\frac{d}{dx}\left(\frac{dy}{dx}\right) = x^2\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{du}\right)$	M1
	$= x^2\left(-\frac{1}{x^2}\frac{dy}{du} + \frac{1}{x}\frac{d}{du}\left[\frac{dy}{du}\right]\frac{du}{dx}\right)$	
	$= -\frac{dy}{du} + x\frac{d^2y}{du^2}.\frac{1}{x} = \frac{d^2y}{du^2} - \frac{dy}{du}$ **(AG)**	A1 [3]
	Substituting in differential equation:	
	$\frac{d^2y}{du^2} - \frac{dy}{du} + 3\frac{dy}{du} + 17y = 34u + 21 \Rightarrow \frac{d^2y}{du^2} + 2\frac{dy}{du} + 17y = 34u + 21$ **(AG)**	B1 [1]
	$m^2 + 2m + 17 = 0 \Rightarrow m = -1 \pm 4i \Rightarrow y_c = e^{-u}\left(A\cos 4u + B\sin 4u\right)$	M1A1
	$y_p = cu + d \Rightarrow y' = c$ and $y'' = 0 \Rightarrow 2c + 17cu + 17d \equiv 34u + 21$ $\Rightarrow c = 2 \quad , \quad d = 1 \Rightarrow y_p = 2u + 1$	M1 A1
	$\Rightarrow y = \frac{1}{x}\left(A\cos[4\ln x] + B\sin[4\ln x]\right) + 2\ln x + 1$ (or in terms of u)	A1√
	$y = 0$ when $x = 1 \Rightarrow 0 = A + 1 \Rightarrow A = -1$ **or** u = 0, y= 0 etc	B1
	$\frac{dy}{dx} = -\frac{1}{x^2}\left(A\cos[4\ln x] + B\sin[4\ln x]\right) + \frac{1}{x}\left(-A\sin[4\ln x].\frac{4}{x} + B\cos[4\ln x].\frac{4}{x}\right) + \frac{2}{x}$	M1
	$\frac{dy}{dx} = -1$ when $x = 1 \Rightarrow -1 = 1 + 4B + 2 \Rightarrow B = -1$ or all in terms of u	A1
	Hence $y = 2\ln x + 1 - \frac{1}{x}\left(\cos[4\ln x] + \sin[4\ln x]\right)$ (oe)	A1 [9]

Qu	Solution	Marks
11 (e)	$\lambda = 1: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -3 & -3 \\ -8 & 5 & -3 \end{vmatrix} = \begin{pmatrix} 24 \\ 24 \\ -24 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ (Or by equations.)	**M1A1**
	$\lambda = 4: \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -3 & -3 \\ -8 & 2 & -3 \end{vmatrix} = \begin{pmatrix} 15 \\ 15 \\ -30 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$	**A1** [3]
	$\begin{pmatrix} 1 & -3 & -3 \\ -8 & 6 & -3 \\ 8 & -2 & 7 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 9 \\ -9 \end{pmatrix} \Rightarrow \lambda = 9$	**M1A1** [2]
	$\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & -2 & -1 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}$	**B1B1**
	Det **P** =1	**B1**√
	Adj $\mathbf{P} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix} \Rightarrow \mathbf{P}^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ -1 & 1 & 0 \end{pmatrix}$. (Or by row operations)	**M1A1**√ [5]
	$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$	**B1**
	$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{C}^2 \Rightarrow \text{A} = \mathbf{P}\mathbf{C}^2\mathbf{P}^{-1} = \mathbf{P}\mathbf{C}\mathbf{P}^{-1}.\ \mathbf{P}\mathbf{C}\mathbf{P}^{-1}$ (i.e. $\mathbf{B} = \mathbf{P}\mathbf{C}\mathbf{P}^{-1}$)	**M1**
	$\mathbf{PC} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 3 \\ -1 & -4 & -3 \end{pmatrix} \Rightarrow \mathbf{B} = \begin{pmatrix} 1 & -1 & -1 \\ -2 & 2 & -1 \\ 2 & 0 & 3 \end{pmatrix}$ (Intermediate step required.)	**M1A1** [4]
	(Note: **B** may not be unique.)	

Qu	Solution	Marks
11 (o) (i)	Direction perpendicular to AB and CD: $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 4 & 3-\lambda & 1 \end{vmatrix} = \begin{pmatrix} \lambda - 2 \\ 4 \\ -4 \end{pmatrix}$	**M1A1**
	$\overrightarrow{DB} = \begin{pmatrix} 5 \\ -4 \\ -6 \end{pmatrix}$. Hence $\left\lvert \dfrac{\begin{pmatrix} 5 \\ -4 \\ -6 \end{pmatrix} . \begin{pmatrix} \lambda - 2 \\ 4 \\ -4 \end{pmatrix}}{\sqrt{(\lambda - 2)^2 + 16 + 16}} \right\rvert = 3$ or equivalent	**M1A1**
(ii)	$\Rightarrow (5\lambda - 2)^2 = 9(\lambda^2 - 4\lambda + 36)$ $\Rightarrow \ldots \Rightarrow \lambda^2 + \lambda - 20 = 0$ (**AG**)	**M1M1** **A1** [7]
	$(\lambda + 5)(\lambda - 4) = 0 \Rightarrow \lambda = -5, 4$	**B1**
	$\lambda = 4 \Rightarrow \mathbf{a} = \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} \Rightarrow$ Normal to $ABD = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & 1 \\ -1 & 3 & 7 \end{vmatrix} = \begin{pmatrix} -10 \\ -29 \\ 11 \end{pmatrix}$	**M1A1**
	$\lambda = -5 \Rightarrow \mathbf{a} = \begin{pmatrix} 2 \\ -5 \\ -3 \end{pmatrix} \Rightarrow$ Normal to $ABD = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 8 & 1 \\ -1 & 12 & 7 \end{vmatrix} = \begin{pmatrix} 44 \\ -29 \\ 56 \end{pmatrix}$	**A1**
	$\cos\theta = \dfrac{-440 + 841 + 616}{\sqrt{10^2 + 29^2 + 11^2}\sqrt{44^2 + 29^2 + 56^2}} = \dfrac{1017}{\sqrt{1062}\sqrt{5913}}$	**M1A1**
	$\Rightarrow \theta = 66.1°$	**A1** [7]