FURTHER MATHEMATICS

Paper 9231/11

Paper 11

Key Messages

- Candidates need to be careful to eliminate arithmetic and numerical errors in their answers.
- Candidates need to read the questions carefully and answer as required.
- Candidates need to avoid sign errors.

General Comments

The scripts for this paper were of an extremely good standard. There were many high quality scripts, as well as many showing evidence of sound learning. Work was well presented by the vast majority of candidates. Solutions were set out in a clear and logical order. The standard of numerical accuracy was good. Algebraic manipulation, where required, was also of a good standard. The work on proof by mathematical induction was pleasingly good. There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. In fact, it was noticeable that candidates, who failed to get a displayed result at their first attempt, had sufficient time to rectify their work. A very high proportion of scripts had substantial attempts at all eleven questions. There were few misreads and few rubric infringements.

Candidates displayed a good knowledge of most topics on the syllabus. There was little evidence of weakness on any particular syllabus topic.

Comments on specific questions

Question 1

There were many correct answers to both parts of this question. The second part caused more difficulty to some candidates. It is worth mentioning that a small minority of candidates had difficulty with squaring and square rooting powers of e, which was somewhat surprising at this level. A small number of candidates were not able to recall the appropriate formula for arc length in polar form.

Answers: **(i)** $\mathrm{e}^{\pi} - \mathrm{e}^{\frac{\pi}{3}}$ (= 20.3), **(ii)** $2\sqrt{2}\left[\mathrm{e}^{\frac{\pi}{2}} - \mathrm{e}^{\frac{\pi}{6}}\right]$ (= 8.83) .

Question 2

Virtually all candidates knew the appropriate results for sums and products of the roots of cubic equations and were able to correctly deduce the results for parts **(i)** and **(ii)**. In part **(iii)**, the vast majority multiplied the equation by x^2, substituted each root in turn and then summed, in order to obtain the sum of the fifth powers of the roots, in terms of the sum of the cubes of the roots and the sum of the squares of the roots, from which they could deduce the required result. Those who sought to obtain a formula for the sum of the fifth powers, or who tried to use a substitution method, made little or no progress.

Question 3

This question proved somewhat more difficult to answer completely than many others on the paper. There was no difficulty in correctly finding S_1, S_2, S_3, S_4, for all but the weakest candidates. The vast majority were then able to establish the result for u_r in terms of r. Only the very best candidates were able to get the mark for justifying the result. The final part was well done by many, but errors occurred due to incorrect use of brackets and using the wrong values for n in the formula given for S_n in the question.

Answers: 3, 10, 21, 36; $\mu_r = 4r - 1$; $6n^2 + n$.

Question 4

Establishing the reduction formula caused difficulty for a good many candidates. Most made an appropriate start, by integrating by parts. In order to make progress, it was necessary to realise that $\sqrt{1+2x} = \frac{1+2x}{\sqrt{1+2x}}$, or equivalent. Those who managed to overcome that hurdle were usually able to complete the derivation of the reduction formula. A small minority differentiated $x(1+x^2)^n$ with respect to x, then integrated the result between 0 and 1 in order to establish the reduction formula. By this method, it is necessary to write x^2 in the form $(1 + x^2 - 1)$. Most candidates were able to show that either $I_0 = \sqrt{3} - 1$ or $I_1 = \frac{1}{3}$ by direct integration, and then use the reduction formula to obtain the printed result for I_3. Since the result was given, complete working was expected.

Question 5

The first three derivatives of the given expression were generally found correctly. Occasionally the second derivative had an incorrect constant term, but the third mark could still be earned if it was differentiated correctly. The proofs by mathematical induction were very satisfactory. Hardly any candidates were put off by the base case being H_3. Most candidates were able to state a correct conclusion involving the use of the word 'integer'. The use of 3, 4, 5,… was accepted.

Answer: $y = \frac{2}{1+x}$.

Question 6

There were many good and accurate attempts to reduce the matrix **M** to echelon form. Most candidates stated the rank of **M** correctly. Without exception candidates formed a system of linear equations, which were solved in order to find a basis for the null space of T. In the final part, marks were lost for lack of generality, where it was required to show that every solution of the given equation was of the given form, where $\{e_1, e_2\}$ was *any* basis for *K*, and not just the one that the candidate had happened to find.

Answers: Rank of **M** = 2, Basis is: $\left\{\begin{pmatrix}-5\\-2\\1\\0\end{pmatrix}, \begin{pmatrix}7\\3\\0\\1\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}1\\0\\-3\\2\end{pmatrix}, \begin{pmatrix}0\\1\\7\\5\end{pmatrix}\right\}$; $\mathbf{M}\begin{pmatrix}1\\-2\\-3\\-4\end{pmatrix} = \begin{pmatrix}2\\16\\10\\22\end{pmatrix}$.

Question 7

The introductory proof was generally well done by most candidates. Weaker candidates, who could recall the result $\mathbf{Ae} = \lambda\mathbf{e}$ were then unable to use it appropriately in order to obtain the result $\mathbf{A}^2\mathbf{e} = \lambda^2\mathbf{e}$. The majority of candidates found the characteristic equation for the matrix **B** and solved it correctly to obtain the eigenvalues of **B**. In the final part of the question there were many correct solutions, but a relatively small number of candidates were not able to deal with the presence of the identity matrix correctly.

Answers: –2, 1, 4; 27, 6, 291.

Question 8

The work on this question by the vast majority of candidates was extremely accurate. In the first part of the question, the errors that did occur were either with the coefficient of **j** in the normal vector, or the value of the constant in the cartesian equation of π_1. In the second part, the main source of any error was caused by confusing sine and cosine, leading to the complement of the correct answer being given as the answer to the problem. In the final part occasional inaccuracies occurred in calculating either the direction vector of the line of intersection of the planes, or in calculating a point on that line. A correct method of solution of all three parts was frequently seen.

Answers: $x + 3y - z = 12$; 75.7° or 1.32 rad.; r = 6i + 2j + t(2i –3j –7k) (OE).

Question 9

The first part of this question was done well by a large number of candidates. There were a small number who thought that they needed $\int 2\pi y \, dx$, but all others attempted to find $\frac{ds}{dt}$ and hence $\int 2\pi y \frac{ds}{dt} dt$. A few had problems simplifying their expression for $\frac{ds}{dt}$, and some others were inaccurate in evaluating the correct integral. The second part proved impossible for a significant minority who found $\int y \, dt$, $\int xy \, dt$ and $\frac{1}{2}\int y^2 dt$, all to no avail. Those who found the appropriate integrals worked extremely accurately and produced the rather awkward coordinates of the centroid, which they would be able to check on their calculators. Pleasingly they realised that answers directly from calculators do not receive credit, so they showed the necessary working to obtain full marks.

Answers: $\frac{11}{9}\pi$ or 3.84; $\left(\frac{4}{7}, \frac{55}{192}\right)$ or (0.571, 0.286).

Question 10

Despite being algebraic, part **(i)** of the question was done well by almost all candidates, with very few marks being lost. Many candidates could not find the correct value for p in part **(ii)**, often because they confused expressions for y and $\frac{dy}{dx}$. Those who obtained the correct value frequently scored most of the marks for this part, dropping only the occasional mark for omitting important features of their graph, or drawing a branch of the curve which clearly did not behave asymptotically. Part **(iii)** was done much better and many candidates were able to show that the gradient function was always positive, hence there were no turning points in the case of $p = 1$. Most could find the points where the curve intersected the x-axis and graphs were generally accurate, again with the occasional mark being lost for wrong forms at infinity.

Answers: **(i)** $x = -1$, $y = px + 4 - p$; **(ii)** $p = 4$; **(iii)** $\left(-2 \pm \sqrt{3},\ 0\right)$.

Question 11 Either

There were very few attempts at this question. Those that were seen were of a generally high standard. All who attempted it gained the first 6 marks. A small number stopped at this stage, not realising that the sixth degree equation was quadratic in x^3. Those who realised this were able to find the three appropriate conjugate pairs and use the principle from the first part of the question to find the required real factors.

Answers: $\cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}\ (k = 0, 1, 2, 3, 4)$; $x^2 - 2\cos\frac{2\pi}{5}x + 1$;

$\left(x^2 - 2\cos\frac{2\pi}{5} + 1\right)\left(x^2 - 2\cos\frac{4\pi}{5} + 1\right)(x - 1)$;

$\cos\frac{\pi}{9} \pm i\sin\frac{\pi}{9}$, $\cos\frac{7\pi}{9} \pm i\sin\frac{7\pi}{9}$, $\cos\frac{13\pi}{9} \pm i\ \sin\frac{13\pi}{9}$;

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Question 11 Or

This much more popular alternative was done extremely well by those who chose it. The first part of the question was done well by many candidates. Some who immediately wrote $y = v^{\frac{1}{2}}$ made the task more complicated for themselves, but often negotiated the task well. The majority wrote $v' = 3y^2 \frac{\mathrm{d}y}{\mathrm{d}x}$ and $v'' = 6y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 3y^2 \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$, from which they were able to show that the two differential equations were equivalent, with little trouble. The complementary function and particular integral were usually accurately found, with only a few candidates making sign or coefficient errors. One arbitrary constant was easily found from the initial conditions, but the second proved troublesome for a significant minority of candidates, often because they were unsure whether they were working with *v* or *y*. The final mark was also lost, from time to time, by those who gave a final expression for *v*, rather than for *y*, as requested in the question. Those giving an expression for y^3 were not penalised on this occasion.

Answer: $y = \{5e^{3x} + 3xe^{3x} + 3e^{-2x}\}^{\frac{1}{3}}$.

FURTHER MATHEMATICS

Paper 9231/12

Paper 12

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There were very few attempts at this question. Those that were seen were of a generally high standard. All who attempted it gained the first 6 marks. A small number stopped at this stage, not realising that the sixth degree equation was quadratic in x^3. Those who realised this were able to find the three appropriate conjugate pairs and use the principle from the first part of the question to find the required real factors.

Answers: $\cos\frac{2k\pi}{5} + \mathrm{i}\sin\frac{2k\pi}{5}\ (k = 0, 1, 2, 3, 4)$; $x^2 - 2\cos\frac{2\pi}{5}x + 1$;

$\left(x^2 - 2\cos\frac{2\pi}{5} + 1\right)\left(x^2 - 2\cos\frac{4\pi}{5} + 1\right)(x - 1)$;

$\cos\frac{\pi}{9} \pm \mathrm{i}\sin\frac{\pi}{9}$, $\cos\frac{7\pi}{9} \pm \mathrm{i}\sin\frac{7\pi}{9}$, $\cos\frac{13\pi}{9} \pm \mathrm{i}\ \sin\frac{13\pi}{9}$;

$\left(x^2 - 2\cos\frac{\pi}{9}x + 1\right)\left(x^2 - 2\cos\frac{7\pi}{9} + 1\right)\ \left(x^2 - 2\cos\frac{13\pi}{9} + 1\right)$.

Question 11 Or

This much more popular alternative was done extremely well by those who chose it. The first part of the question was done well by many candidates. Some who immediately wrote $y = v^{\frac{1}{2}}$ made the task more complicated for themselves, but often negotiated the task well. The majority wrote $v' = 3y^2 \frac{dy}{dx}$ and $v'' = 6y\left(\frac{dy}{dx}\right)^2 + 3y^2 \frac{d^2y}{dx^2}$, from which they were able to show that the two differential equations were equivalent, with little trouble. The complementary function and particular integral were usually accurately found, with only a few candidates making sign or coefficient errors. One arbitrary constant was easily found from the initial conditions, but the second proved troublesome for a significant minority of candidates, often because they were unsure whether they were working with v or y. The final mark was also lost, from time to time, by those who gave a final expression for v, rather than for y, as requested in the question. Those giving an expression for y^3 were not penalised on this occasion.

Answer: $y = \{5e^{3x} + 3xe^{3x} + 3e^{-2x}\}^{\frac{1}{3}}$.

FURTHER MATHEMATICS

Paper 9231/13

Paper 13

Key Messages

- Candidates need to be careful to eliminate arithmetic and numerical errors in their answers.
- Candidates need to read the questions carefully and answer as required.
- Candidates need to avoid sign errors.

General Comments

The standard of work seen was generally of a high quality and some candidates made excellent responses to the questions. Solutions were set out in a clear and logical order. The standard of numerical accuracy was good. Algebraic manipulation, where required, was also of a high standard. Calculus and vector work, in particular, were very impressive.

There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. All of the scripts had substantial attempts at all eleven questions. There were no misreads or rubric infringements seen.

Candidates displayed a sound knowledge of most topics on the syllabus. As well as the calculus and vector work, already mentioned, candidates tackled the questions on series, roots of equations and rational functions confidently. The majority of the candidates were able to cope well with the questions on complex numbers, proof by induction and polar curves, which are often done less well.

Comments on specific questions

Question 1

Most candidates were able to find the partial fractions correctly and apply them to summing the series. There were a few cases with incorrect cancellation of terms, resulting in an incorrect sum. The final mark, for the sum to infinity, could be earned when the candidate followed through correctly from an incorrect sum.

Answers: $\dfrac{1}{r(r-1)(r+1)} = \dfrac{1}{2(r-1)} - \dfrac{1}{r} + \dfrac{1}{2(r+1)}$; $\dfrac{1}{4} - \dfrac{1}{2n} + \dfrac{1}{2(n+1)}$ (OE); $S_\infty = \dfrac{1}{4}$.

Question 2

Most candidates were able to show that the determinant of the given matrix was zero. The occasional candidate failed to state what this implied for the second mark. In the second part of the question a small number of candidates only gave a point which was in all three planes, without giving the general solution. The majority, however, gave a correct answer. The occasional candidate omitted '**r** =' in their answer.

Answer: $r = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ (OE).

Question 3

All of the candidates were familiar with solving a second order linear differential equation, and very few errors occurred.

Answer: $y = e^{-x}\left(A\cos\sqrt{3}x + B\sin\sqrt{3}x\right) + x^2 - x + 2$.

Question 4

Most candidates were able to complete both parts of this question correctly. All candidates knew what to do. A small minority made the occasional sign error, or confused sines and cosines, in their working.

Answer: – 1.

Question 5

This question was understood well by all candidates. The basic results about sums and products of roots of a cubic equation were quoted correctly. Only the occasional sign or algebraic error occurred in the working of some candidates.

Answers: Roots: $-\frac{7}{2}, -\frac{3}{2}, \frac{1}{2}$; k = 22.

Question 6

Part **(a)** was done less well than part **(b)** by a small number of candidates who could not recall the appropriate formula for the mean value of *y* with respect to *x*. The majority of candidates were able to do both parts correctly.

Answers: **(a)** $\frac{6}{\pi}\ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right)$ (OE) or 1.47 (3sf); **(b)** $\ln(2+\sqrt{3})$ or 1.32 (3sf).

Question 7

This question was done well by most of the candidates. Those who expressed the equation of the curve in the form $y = 2x + 1 – 3(x + 2)^{-1}$ in order to determine the oblique asymptote had an easier task in the second part, since the form $y' = 2 + 3(x+2)^{-2}$ led immediately to the required result. The sketches were generally good, but the occasional candidate had a curve whose branches never approached the asymptotes sufficiently well, which forfeited one mark.

Answers: x = – 2, y = 2*x* + 1.

Question 8

Most candidates scored well on this question. The first part invariably produced the correct answer, with only the occasional candidate producing an arithmetical error, either in the sign of the **j** component of the vector product, or in calculating the constant term. Similarly the second part was usually correct apart from the odd candidate who failed to solve 3*t* = 1 correctly, thus getting incorrect coordinates for *E*, which had a knock on effect in the final part of the question. In the final part the main source of any error was where a candidate used cosine rather than sine and obtained the complement of the correct answer. There were a pleasing number of completely correct solutions to this vector question, indicating that the candidates had been well taught and had learnt the material well.

Answers: 6*x* + 2*y* – 5*z* = 4; (2, 1, 2); 9.5° (0.166 rad.).

Question 9

The bookwork at the start of this question, which was also a test of mathematical induction, was generally done well, with only the occasional candidate having difficulty in using the appropriate trigonometric formula. Most candidates were able to derive the correct expression for $\sin^5\theta$. A number of candidates, however, made errors with signs, usually associated with powers of i. The occasional incorrect binomial expansion was also seen.

Answer: $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

Question 10

In the first part of this question, candidates realised that they needed to find where $\frac{\mathrm{d}r}{\mathrm{d}\theta}=0$. Most managed to do this correctly, but there were some incorrect quadratic equations in $\cos\theta$ and some incorrect solutions when the correct quadratic equation had been obtained. A final point on this first part is to note that giving the polar coordinates as (θ, r) is not acceptable. Sketches were generally of the correct shape with the occasional one incorrectly orientated. There was also the occasional example with an extra loop, presumably where the candidate had set the domain as 0 to 2π on their calculator. In the final part, the correct formula was invariably used to determine the area of a sector, and the correct limits were also used for the required region. There were few errors made in the integration. These were mostly due to an incorrect double angle formula being used, or the occasional sign error.

Answers: $\left(\frac{3}{2}\sqrt{3}, \frac{2}{3}\pi\right)$; $\frac{5}{16}\pi - \frac{1}{2} - \frac{\sqrt{2}}{3}$.

Question 11 Either

Those selecting this alternative usually derived the reduction formula by integrating by parts. The occasional candidate adopted the alternative strategy of differentiating $x(1 + x^2)^n$ and then integrating the result, with respect to x between 0 and 1. In either approach, it is necessary to express x^2 in the form $(1 + x^2 - 1)$, and most candidates were aware of this technique. Since the answer was a displayed result, it was necessary to show complete working. Pleasingly this was the case for those who attempted this alternative. The second part of the question was invariably done well. The final part proved a little more troublesome for a small number of candidates.

Answers: $I_0 = 1$, $I_3 = \frac{96}{35}$ (2.74); $I_{-3} = \frac{3}{32}\pi + \frac{1}{4}$.

Question 11 Or

Those selecting this alternative were invariably able to quote $\mathbf{Ae} = \lambda\mathbf{e}$ and $\mathbf{Be} = \mu\mathbf{e}$. The next two lines of the initial proof caused difficulty for a few of the candidates. The condition that $\mathbf{e} \neq \mathbf{0}$ was not insisted upon, but should be noted, as an example of good practice. The correct eigenvalues of the matrix A were obtained by virtually all doing this alternative, who were then able, mostly, to find the correct eigenvalues of the matrix **M**. Most candidates were able to state the correct matrix **R** and recognise that the matrix **S** was $\mathbf{R}^{-1}$. Some errors occurred in determining $\mathbf{R}^{-1}$. These were either sign errors, or an incorrect value for the determinant of **R**.

Most candidates found the correct diagonal matrix **D**, apart from those who had incorrect eigenvalues earlier, or who made a slip with $(-8)^5$.

Answers: –1, 1, 3; –8, –2, 2; $\mathbf{R} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, $\mathbf{S} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 2 \\ -1 & -1 & 2 \\ 2 & 0 & -2 \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} -32768 & 0 & 0 \\ 0 & -32 & 0 \\ 0 & 0 & 32 \end{pmatrix}$.

FURTHER MATHEMATICS

Paper 9231/21

Paper 21

Key messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

General comments

Almost all candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 11**, there was a preference for the Statistics option, though the minority of candidates who chose the Mechanics option frequently produced equally good attempts. Indeed all questions were answered well by some candidates, with **Questions 1, 3 and 8** found to be the most challenging.

Advice to candidates in previous reports to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable was seemingly heeded by many. When relevant in Mechanics questions it is helpful to include diagrams which show, for example, what forces are acting and also their directions as in **Question 4** and the directions of motion of particles, as in **Question 5**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. This is also true in Statistics questions, and thus in **Questions 7 and 9** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

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Comments on specific questions

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While almost all candidates found a moment of inertia for each of the three rods and the disc and then summed them to produce the moment of inertia for the system, very few of these attempts were completely correct. Candidates need to consider carefully the extent to which the cases in the *List of Formulae* are immediately applicable here and whether any applications of the parallel and/or perpendicular axes theorems are required. Thus for AB and CD, the parallel axes theorem is used to effectively transfer the relevant axis from their centres to their ends A and D, giving a moment of inertia for each about the axis AD

of $4Ma^2/3$. By contrast neither the formula for the moment of inertia of a rod nor the parallel axis theorem is needed for *BC*, since its moment of inertia is effectively that of a particle of mass *M* about an axis at a distance $2a$ away, giving a result of $4Ma^2$. As for the disc, the *List of Formula* yields a moment of inertia $Ma^2/3$ about the axis through its centre and perpendicular to its plane, so use of the perpendicular axis theorem gives a moment of inertia about its diameter *AD* of $Ma^2/6$. In any event it is inadvisable to provide an attempt consisting solely of the addition of four or more terms with no explanation whatever, particularly when their summation yields an incorrect final answer, since credit may be needlessly lost through lack of clarity.

Answer: $(41/6)\,Ma^2$.

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The first part of this question was almost always answered correctly, with the standard SHM formula $T = 2\pi/\omega$ yielding $\omega = 2$, followed by two applications of another SHM formula $v^2 = \omega^2 (A^2 - x^2)$ to first find $A^2 = 45$ and then the value of v when $x = 6$. The required time is also found from two applications of an SHM formula such as $x = A \sin \omega t$, with $x = 3$ and 6 respectively. It is important that the argument of the trigonometric function be in radians and not degrees, since otherwise any times found in this way will be incorrect by a factor $180/\pi$.

Answer: 6 ms^{-1}; 0·32 s.

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This was found to be one of the most challenging questions on the paper, with only a minority of candidates using a valid approach. One method of solution is to first use the given value of the angular speed ω when the disc has turned through 2 radians to find its angular acceleration $d\omega/dt = 25/4$ rad s^{-2}, or equivalently the linear acceleration 5/4 m s^{-2} of the block. Equating the couple on the disc due to the tension *T* N of the string to the product of the moment of inertia of the disc and its angular acceleration then yields *T*. Finally the application of Newton's second law of motion to the block, which is descending in a straight line and is subject to a net downward force of $4\,g - R - T$, enables *R* to be found. An alternative but equally valid approach may be based on conservation of energy, in which case the linear or angular acceleration need not be found explicitly. Some candidates appeared to treat the whole system as a rigid body, equivalent to the block being attached to the rim of the disc, effectively increasing the moment of inertia of the disc by $4 \times 0{\cdot}2^2$. While this approach can produce the correct answers, none of the candidates using it attempted to justify it or even adequately explain what they were doing, and it is arguably not within the syllabus so should be used with caution.

Answer: $R = 135/4$; $T = 5/4$.

Question 4

Many successful attempts were seen at this question, usually first employing moments about *O* to relate *P* to the frictional forces at *A* and *B*. Two further moment or force resolution equations will then give a total of three independent equations involving *P* and the limiting friction and normal reaction at both *A* and *B*. Since the coefficient of friction is known to be ½ at both these points, the number of unknown forces reduces to three and so the three equations may be solved in a variety of ways to produce the given expression for *P*. Solving for the other two unknowns, whether friction or reaction forces, provides the required ratio in the final part. The meaning of symbols used for the unknown forces is clearer if candidates either choose self-evident symbols such as F_A for the limiting friction at *A*, or else provide a diagram in their answer booklet or paper with the forces shown on it. They should be aware that anything they write on the question paper will not be seen by the Examiners.

Answer: **(ii)** 7/8.

Question 5

Solution of the two equations resulting from conservation of momentum and Newton's restitution equation gives the speeds 4 $u/9$ and 10 $u/9$ of the two particles after the first collision, and that of B is changed by a factor e by its collision with the barrier, with its direction reversed. A further application of conservation of momentum and Newton's restitution equation to the second collision between the particles similarly produces two equations for the velocities v_A and v_B after this collision. Replacing v_B by $5v_A$, for example, enables v_A to be eliminated and a value of e found. Not all candidates realised that the second value of e required by the question results from taking instead $v_B = -5v_A$.

Answer: 2/13, ½.

Question 6

The mean value of X was usually stated correctly, and the probability that obtaining a 5 or a 6 takes more than 8 throws was often found by correctly calculating q^8 with $q = ⅔$. The final part is more challenging, requiring the least value of the integer n which satisfies $1-q^{n-1} > 0{\cdot}99$ rather than $1-q^n > 0{\cdot}99$.

Answer: 3; 256/6561 or 0·0390; 13.

Question 7

As in all such tests, the hypotheses should be stated in terms of the population mean and not the sample mean. The unbiased estimate 0·9422 of the population variance may be used to calculate a t-value of magnitude 1·50. Since it is a one-tail test, comparison with the tabulated value of 1·383 leads to acceptance of the alternative hypothesis, namely that the population mean is less than 7·5.

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Simply assuming that the lifetime of the electrical component has a negative exponential distribution in order to deduce that $A = \lambda$ is not justified by the information given, and instead candidates should recall that the integral of f over the interval $t \geq 0$ must be unity. Similarly it is invalid to then assume that λ can simply be stated as a number such as 0·16, and it must instead be estimated by equating the integral of f(t) over the interval (0, 1) to 0·16 and solving for λ. With this estimate of λ, the median value m of T may then be estimated from $e^{-\lambda m} = ½$.

Answer: **(ii)** 0·174, 3·98.

Question 9

The product moment correlation coefficient r for the sample was found correctly by almost all candidates, being the positive square root of the product of the gradients 4·21 and 0·043. The test requires an explicit statement of the null and alternative hypotheses, $\rho = 0$ and $\rho \neq 0$, and here candidates should be aware that r and ρ are not the same entity. Comparison of the magnitude of the previously calculated value of r with the tabular value 0·549 leads to a conclusion of there being no evidence of non-zero correlation. The required mean values are found by recalling that these values satisfy both of the regression line equations, which may therefore be solved as simultaneous equations. Finally most candidates successfully solved one or less often both of the equations of the regression lines for x when $y = 2{\cdot}3$, and while they often stated that their answer was unreliable, the reason given varied. There is no single acceptable reason, but the two seen most frequently were the lack of evidence of non-zero correlation and the differing values of x given by the two regression line equations.

Answers: **(i)** 0·425; **(iii)** 7·72, 31·6; **(iv)** 0·751 or 6·46.

Question 10

Most candidates produced good answers to this question, particularly the first part, though a few did not realise that the appropriate test is a chi-squared one. After stating the hypotheses to be tested, a table of the expected numbers of customers is produced in the usual way and preferably to an accuracy of two (or more) decimal places. The calculated χ^2-value 1·64 (or 1·61 if the expected numbers were found to only one decimal place) should be compared with the tabular value 5·991, leading to acceptance of the null hypothesis. Thus there is no difference between the coffee preferences of the male and female customers. The key to the second part is to realise that the calculated value 1·64 of χ^2 increases by a factor n, while the critical value 5·991 is unaltered. Thus we require the smallest integer n satisfying $n > 5{\cdot}991/1{\cdot}64 \approx 3{\cdot}7$.

Answer: 4.

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While less popular than the Statistics alternative discussed below, many of those who chose this optional question did, however, make good attempts at the first part. This involves combining conservation of energy, which gives an expression for the speed when OP makes an angle θ with the upward vertical, with an application of Newton's second law of motion in a radial direction to the particle P. Taking the normal reaction to be zero when P loses contact with the sphere yields the given value of cos θ. In the second part candidates need to consider the vertical component of the subsequent motion of P, rather than wrongly assuming that it will effectively continue to move in the same direction until striking the plane. The correct approach is to find the vertical component of the speed on leaving the sphere, and hence just before striking the plane after falling under gravitational acceleration. The particle then moves upwards with the vertical component of its speed changed by a factor 5/9 in the collision, and hence the height risen may be found.

Answer: $3a/5$.

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After stating the hypotheses, which should be in terms of the population means and not the sample means, most candidates found unbiased estimates s_x^2 and s_y^2 of the population variances using X's and Y's sample respectively, and then an estimate s^2 of the population variance for the combined sample using $s^2 = s_x^2/60 + s_y^2/80$. The magnitude of the z–value required for the test is then $(1{\cdot}22 - 0{\cdot}97)/s$ which is approximately 1·80. Given the large size of the samples, it is acceptable here to use biased estimates instead, leading to 1·81. A minority of candidates made the assumption, which is acceptable provided it is stated explicitly, of equal population variances and used a pooled estimate of this common variance to produce a z-value of magnitude 1·73. In any event, finding the set of possible values of α proved to be more challenging. The table of the Normal distribution function should be used to find, for example, $\Phi(1{\cdot}80) = 0{\cdot}9641$ or equivalently 96·41%, implying that $\alpha > 3{\cdot}59$ though one decimal place, or even an integral value 4, is probably sufficient accuracy in such a result. Similarly a z-value of 1·73 leads to $\alpha > 4{\cdot}18$.

Answer: $\alpha > 3{\cdot}6$.

FURTHER MATHEMATICS

Paper 9231/22

Paper 22

Key messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

All steps of the argument or derivation should be written down in those questions which require a given result to be shown. Although less detail may suffice in questions where an unknown value or result is to be found, candidates are advised to show sufficient working so that credit may be earned for their method of solution even if an error occurs.

General comments

Almost all candidates attempted all the questions, and while very good answers were frequently seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 11**, there was a preference for the Statistics option, though the minority of candidates who chose the Mechanics option frequently produced equally good attempts. Indeed all questions were answered well by some candidates, with **Questions 1, 3 and 8** found to be the most challenging.

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Answer: 4.

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While less popular than the Statistics alternative discussed below, many of those who chose this optional question did, however, make good attempts at the first part. This involves combining conservation of energy, which gives an expression for the speed when OP makes an angle θ with the upward vertical, with an application of Newton's second law of motion in a radial direction to the particle P. Taking the normal reaction to be zero when P loses contact with the sphere yields the given value of cos θ. In the second part candidates need to consider the vertical component of the subsequent motion of P, rather than wrongly assuming that it will effectively continue to move in the same direction until striking the plane. The correct approach is to find the vertical component of the speed on leaving the sphere, and hence just before striking the plane after falling under gravitational acceleration. The particle then moves upwards with the vertical component of its speed changed by a factor 5/9 in the collision, and hence the height risen may be found.

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Answer: $\alpha > 3{\cdot}6$.

FURTHER MATHEMATICS

Paper 9231/23

Paper 23

Key messages

To score full marks in the paper candidates must be well versed in both Mechanics and Statistics, though any preference between these two areas can be exercised in the choice of the final optional question.

General comments

The majority of the candidates attempted all the questions. In the only question which offered a choice, namely **Question 11**, there was a slight preference for the Statistics option, and in general this produced rather better attempts than the Mechanics one.

Comments on specific questions

Question 1

The radial component of the particle's acceleration may be readily found from the standard formula v^2/r, while the transverse component is the derivative of the given expression for the speed v, with $t = 3$ in both cases. There is no need to involve angular speed or acceleration.

Answer: 50 m s^{-2}; 4 m s^{-2}.

Question 2

Both parts of this question were generally well done, with each requiring a combination of conservation of momentum and Newton's restitution equation.

Answer: $\lambda = 2$.

Question 3

Two equations must be formulated at each point A and B, based on conservation of energy and on equating the net radial force to mv^2/a. Elimination of T and the speeds at A and B then yields the required expression for u, though this process can be performed in a variety of ways.

Answer: $u = \sqrt{(5\,ga)}$.

Question 4

Equating forces at the equilibrium point shows that the modulus of elasticity is 4 mg, and the application of Newton's Law at a general point produces the standard SHM equation from which the period may be found by the usual formula $2\pi/\omega$. Applying another standard formula $v^2 = \omega^2 (A^2 - x^2)$ at both the mid-point of the motion and an unknown point at which the speed is one-half the maximum yields the required equal distances of the latter from the mid-point and hence the distances from O.

Answer: $2\pi\sqrt{(a/g)}$; $(5 \pm \frac{1}{4}\sqrt{3})\,a$.

Question 5

One possible method of solution is to find the reaction at Q by resolving vertically for the disc, while the given reaction at P may be verified by either resolving horizontally or taking moments about Q for the disc. The given reaction at B then follows by, for example, taking moments about A for the rod. In the final part, candidates should not assume that the reaction at the hinge A is vertical, and instead should find its vertical and horizontal components before combining them.

Answer: $k = 3{\cdot}02$.

Question 6

This question was generally well answered. For completeness, candidates should state the distribution function of T for $t < 0$ as well as for $t \geq 0$.

Answer: 5; $1 - e^{-t/5}$ $(t \geq 0)$, 0 otherwise; 0·135.

Question 7

Most candidates correctly equated the expression for the two-sample estimate of the common variance given in the *List of Formulae* to the specified value 2, and selected the integral root of the resulting quadratic equation in n.

Answer: 5.

Question 8

Most candidates produced good answers to this question, apart from not always using the tabular value with the appropriate degrees of freedom. Provided the last four cells are combined to ensure the expected values are no less than 5, the calculated χ^2-value 7·18 (or 7·14 if the expected values are rounded to only one decimal place) should be compared with the tabular value 5·991, leading to rejection of the null hypothesis. Thus the binomial distribution B(6, ¼) does not fit the data.

Question 9

As in all such tests, the hypotheses required in the first part of this question should be stated in terms of the population mean and not the sample mean. The unbiased estimate 0·1687(5) of the population variance may be used to calculate a t-value of 2·19. Since it is a one-tail test, comparison with the tabulated value of 2·306 leads to acceptance of the null hypothesis, namely that the population mean of X is indeed 10·2. The confidence interval in the second part is found from the usual formula incorporating a tabulated value of 1·86, and centred on the sample mean rather than 10·2.

Answer: $10{\cdot}5 \pm 0{\cdot}255$.

Question 10

Finding both the product moment correlation coefficient r for the sample and the equation of the required regression line $y = a + bx$ is straightforward, with the relevant expressions given in the *List of Formulae*. Apart from a rare arithmetical error candidates experienced little difficulty. The final test requires an explicit statement of the null and alternative hypotheses, $\rho = 0$ and $\rho \neq 0$, and here candidates should be aware that r and ρ are not the same entity. Comparison of the previously calculated value of r with the tabular value 0·658 leads to a conclusion of there being evidence of non-zero correlation.

Answer: 0·737; 3·26 kg.

Question 11 (Mechanics)

All of the candidates attempting this optional question found the moment of inertia of the body about the axis through *C* correctly. This is then inserted into the equation of circular motion and sin θ approximated by θ to produce the standard form of the SHM equation with $\omega^2 = 11\ g/64a$, from which the given period is verified from the usual formula $2\pi/\omega$. Candidates should note in the second part that the appropriate energy equation involves rotational rather than linear kinetic energy, yielding the maximum angular speed of the body from which the required maximum speed of the point *A* is found by multiplying by its distance from *C*, namely $10a$.

Answer: $\sqrt{(55\ ga/4)}$.

Question 11 (Statistics)

The given form of G(t) may be found from 1 – F(60 – t), with the cumulative distribution function F of X obtained by integrating f(x). Candidates had little difficulty finding the median and mean of T. The former is found in the usual way by solving G(t) = ½, and after finding the probability density function g of T by differentiating G(t), the mean of T is found by integrating tg(t) over the interval (0, 60).

Answer: 9·63; 15.

