UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education Advanced Level

FURTHER MATHEMATICS **9231/11**

Paper 1 **May/June 2013**

3 hours

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF10)

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READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **4** printed pages.

UNIVERSITY *of* CAMBRIDGE
International Examinations

[Turn over

1 Find the area of the region enclosed by the curve with polar equation $r = 2(1 + \cos\theta)$, for $0 \leqslant \theta < 2\pi$. [4]

2 Prove by mathematical induction that $5^{2n} - 1$ is divisible by 8 for every positive integer n. [5]

3 The cubic equation $x^3 - 2x^2 - 3x + 4 = 0$ has roots α, β, γ. Given that $c = \alpha + \beta + \gamma$, state the value of c. [1]

Use the substitution $y = c - x$ to find a cubic equation whose roots are $\alpha + \beta$, $\beta + \gamma$, $\gamma + \alpha$. [3]

Find a cubic equation whose roots are $\dfrac{1}{\alpha+\beta}$, $\dfrac{1}{\beta+\gamma}$, $\dfrac{1}{\gamma+\alpha}$. [2]

Hence evaluate $\dfrac{1}{(\alpha+\beta)^2} + \dfrac{1}{(\beta+\gamma)^2} + \dfrac{1}{(\gamma+\alpha)^2}$. [2]

4 Let $I_n = \displaystyle\int_0^1 \frac{1}{(1+x^2)^n}\,dx$. Prove that, for every positive integer n,

$$2nI_{n+1} = 2^{-n} + (2n-1)I_n.$$ [5]

Given that $I_1 = \frac{1}{4}\pi$, find the exact value of I_3. [3]

5 Use the method of differences to show that $\displaystyle\sum_{r=1}^{N} \frac{1}{(2r+1)(2r+3)} = \frac{1}{6} - \frac{1}{2(2N+3)}$. [5]

Deduce that $\displaystyle\sum_{r=N+1}^{2N} \frac{1}{(2r+1)(2r+3)} < \frac{1}{8N}$. [4]

6 The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 4 & -5 & 3 \\ 3 & -4 & 3 \\ 1 & -1 & 2 \end{pmatrix}.$$

Show that $\mathbf{e} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ and state the corresponding eigenvalue. [2]

Find the other two eigenvalues of $\mathbf{A}$. [4]

The matrix $\mathbf{B}$ is given by

$$\mathbf{B} = \begin{pmatrix} -1 & 4 & 0 \\ -1 & 3 & 1 \\ 1 & -1 & 3 \end{pmatrix}.$$

Show that $\mathbf{e}$ is an eigenvector of $\mathbf{B}$ and deduce an eigenvector of the matrix $\mathbf{AB}$, stating the corresponding eigenvalue. [3]

7 By considering the binomial expansion of $\left(z - \frac{1}{z}\right)^6$, where $z = \cos\theta + \mathrm{i}\sin\theta$, express $\sin^6\theta$ in the form

$$\tfrac{1}{32}(p + q\cos 2\theta + r\cos 4\theta + s\cos 6\theta),$$

where p, q, r and s are integers to be determined. [6]

Hence find the exact value of $\displaystyle\int_0^{\frac{1}{4}\pi} \sin^6\theta \,\mathrm{d}\theta$. [4]

8 The linear transformations $\mathrm{T}_1 : \mathbb{R}^4 \to \mathbb{R}^4$ and $\mathrm{T}_2 : \mathbb{R}^4 \to \mathbb{R}^4$ are represented by the matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ respectively, where

$$\mathbf{M}_1 = \begin{pmatrix} 1 & -2 & 3 & 5 \\ 3 & -4 & 17 & 33 \\ 5 & -9 & 20 & 36 \\ 4 & -7 & 16 & 29 \end{pmatrix} \quad \text{and} \quad \mathbf{M}_2 = \begin{pmatrix} 1 & -2 & 0 & -3 \\ 2 & -1 & 0 & 0 \\ 4 & -7 & 1 & -9 \\ 6 & -10 & 0 & -14 \end{pmatrix}.$$

The null spaces of T_1 and T_2 are denoted by K_1 and K_2 respectively. Find a basis for K_1 and a basis for K_2. [6]

It is given that $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$. The vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are such that $\mathbf{M}_1\mathbf{x}_1 = \mathbf{M}_1\mathbf{a}$ and $\mathbf{M}_2\mathbf{x}_2 = \mathbf{M}_2\mathbf{a}$. Given that $\mathbf{x}_1 - \mathbf{x}_2 = \begin{pmatrix} p \\ 5 \\ 7 \\ q \end{pmatrix}$, find p and q. [4]

9 Find x in terms of t given that

$$4\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 4\frac{\mathrm{d}x}{\mathrm{d}t} + x = 6\mathrm{e}^{-2t},$$

and that, when $t = 0$, $x = \frac{5}{3}$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{7}{6}$. [9]

State $\lim\limits_{t\to\infty} x$. [1]

[Questions 10 and 11 are printed on the next page.]

10 The curve C has equation $y = \dfrac{2x^2 - 3x - 2}{x^2 - 2x + 1}$. State the equations of the asymptotes of C. [2]

Show that $y \leqslant \frac{25}{12}$ at all points of C. [4]

Find the coordinates of any stationary points of C. [3]

Sketch C, stating the coordinates of any intersections of C with the coordinate axes and the asymptotes. [4]

11 Answer only **one** of the following two alternatives.

EITHER

The curve C has equation $y = 2\sec x$, for $0 \leqslant x \leqslant \frac{1}{4}\pi$. Show that the arc length s of C is given by

$$s = \int_0^{\frac{1}{4}\pi} (2\sec^2 x - 1)\,\mathrm{d}x. \qquad [4]$$

Find the exact value of s. [2]

The surface area generated when C is rotated through 2π radians about the x-axis is denoted by S. Show that

(i) $S = 4\pi \displaystyle\int_0^{\frac{1}{4}\pi} (2\sec^3 x - \sec x)\,\mathrm{d}x$, [3]

(ii) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sec x \tan x) = 2\sec^3 x - \sec x$. [3]

Hence find the exact value of S. [2]

OR

The points A, B, C and D have coordinates as follows:

$$A\,(2,\ 1,\ -2), \quad B\,(4,\ 1,\ -1), \quad C\,(3,\ -2,\ -1) \quad \text{and} \quad D\,(3,\ 6,\ 2).$$

The plane Π_1 passes through the points A, B and C. Find a cartesian equation of Π_1. [4]

Find the area of triangle ABC and hence, or otherwise, find the volume of the tetrahedron $ABCD$. [6]

[The volume of a tetrahedron is $\frac{1}{3} \times$ area of base $\times$ perpendicular height.]

The plane Π_2 passes through the points A, B and D. Find the acute angle between Π_1 and Π_2. [4]