FURTHER MATHEMATICS

Paper 9231/11
Paper 11

Key messages

In order that candidates receive credit where possible for incorrect answers, they should remember to show full working and reasoning. Incorrect answers without working cannot be given any credit, whereas partial credit can be awarded if a correct method is shown. Where candidates are asked to 'show' or 'prove' a result, marks will be lost if any essential steps in the argument are omitted.

General comments

The scripts for this paper were of a generally good quality. There were a good number of high quality scripts and many showing evidence of sound learning. Work was well presented by the vast majority of candidates. Solutions were set out in a clear and logical order. The standard of numerical accuracy was good. Algebraic manipulation, where required, was of a sound standard. Some of the vector work was of high quality.

There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. There were few misreads and few rubric infringements.

Candidates displayed a sound knowledge of most topics on the syllabus. As well as the vector work, already mentioned, candidates tackled the questions on matrices, summation of series, roots of equations, and graphs of rational functions confidently. The question on proof by induction was a cause of difficulty for some candidates.

Comments on specific questions

Question 1

Using partial fractions with the method of differences was a technique that was well known to the vast majority of candidates. Many obtained the sum in one of several acceptable forms, one of which led nicely to the sum to infinity.

Answers: $\frac{1}{2}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right)$; $\frac{1}{6}-\frac{1}{2(2n+3)}$; $\frac{1}{6}$.

Question 2

Most candidates were able to express $\Sigma\alpha$ and $\Sigma\alpha\beta$ in terms of p and q. They were then able to eliminate k and obtain the required result. Rather fewer were able to use the product of the roots to obtain the relation between p, q and r. Occasional sign errors were the main problem.

Question 3

Many candidates were able to reduce the matrix **M** to echelon form and hence state the rank of **M** correctly. A few stopped short of echelon form.

The second part was less well done. Some candidates wrote down a set of equations and progressed no further. Better candidates were able to solve the equations to find a suitable basis for the null space of T.

Answers: **(i)** 2; **(ii)** $\left\{\begin{pmatrix}-2\\0\\1\\1\end{pmatrix}, \begin{pmatrix}-3\\1\\0\\0\end{pmatrix}\right\}$ (for example).

Question 4

Part **(i)** was invariably done well. In part **(ii)** often the only marks obtained were for stating the inductive hypothesis, H_k, and/or proving the base case for $k = 1$. It was rare to find a rigorous argument for the inductive step. The better candidates argued that assuming $f(k) = 7\lambda$ for some positive integer k and, by part **(i)**, $f(k + 1) + f(k) = 7\mu$ so $f(k + 1) = 7(\mu - \lambda)$, etc. Some proofs also lacked a satisfactory concluding statement.

Question 5

A considerable number of polar graphs had extra loops, since candidates were including points where r was negative. The syllabus specifically says that only cases where r is positive should be taken. The formula for the area of a sector, using polar coordinates was well known, as was the use of the double angle formula in the resulting integral. There was some confusion over the correct limits to use in order to find the area of one loop.

Answer: $\frac{1}{2}\pi$.

Question 6

There were essentially two approaches to this question, which produced some good work. The first method was to find the direction vector of the common perpendicular by use of the vector product, using the direction vectors of each line. The vector between the two given points on l_1 and l_2 was then found and the scalar product of this vector with the unit vector in the direction of the common perpendicular gave the required distance. A number of candidates, finding that the common perpendicular was 3**i** – 4**j** +12**j,** just stated $\sqrt{3^2+(-4)^2+12^2}=13$, which earned no marks. Using the first approach, the usual method for finding **p** and **q** was to write the vector $\overrightarrow{PQ}$ in terms of two parameters and equate to ±t(3**i** – 4**j** +12**k**) and solve three simultaneous equations. Since $|t|$ was 1, those who omitted t sometimes obtained the correct answer; this received no credit. The second method, which was slightly more common, was to write $\overrightarrow{PQ}$ in terms of two parameters and form two equations by forming the scalar product of $\overrightarrow{PQ}$ with the direction vectors of l_1 and l_2. Solving these equations led to **p** and **q**. Fortunately the parameters were integers in this case, had they not have been, the first method would have been an easier calculation. Having obtained **p** and **q**, simple coordinate geometry gave the length PQ.

Answers: **(ii)** 2**j** – 7**k**, 3**i** – 2**j** + 5**k**.

Question 7

The candidates who successfully proved the initial result generally differentiated the given substitution twice and inserted the results in the given equation. This then simplified to the required equation with little algebraic manipulation. Some evidence of manipulation was required, however, as the result was displayed.

Those who rearranged the given substitution to $y = v^{\frac{1}{3}}$, before differentiating twice, made life difficult for themselves and frequently did not complete this part of the question. There were also attempts at this part of the question using various chain rules, which often resulted in an incomplete solution. Good attempts were made at the complementary function and particular integral of the *v*-*x* equation. The final mark was sometimes lost, when candidates did not find *y* in terms of *x*, as requested.

Answer: $y = \left\{Ae^{-5x} + Be^{3x} - \frac{3}{2}e^{-x}\right\}^{\frac{1}{3}}$.

Question 8

The answers to this question were complete and very accurate. The characteristic equation was invariably found and solved correctly. Nearly all candidates had an acceptable method for finding eigenvectors, either by use of equations, or by vector products. Some errors were made in finding eigenvectors. All marks in the final part could be earned on a follow through basis. In this final part errors were usually small slips with a sign, or not finding fifth powers in the matrix **D**

Answers: Eigenvalues: –3, 2, 5; Eigenvectors: $t(\mathbf{j} + \mathbf{k})$, $t(\mathbf{i} + \mathbf{j} - \mathbf{k})$, $t(2\mathbf{i} - \mathbf{j} + \mathbf{k})$;
$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$; $\begin{pmatrix} -243 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 3125 \end{pmatrix}$.

Question 9

This question was also done very accurately and the awkward fractions involved presented little difficulty to the vast majority of candidates. A small number of candidates only found one of the coordinates, either $\bar{x}$ or $\bar{y}$, this earned four of the seven marks available. The final part was also done well and many candidates were able to write down the integral by inspection, without needing to resort to a substitution. Sufficient working, of course, was required, as the answer was given.

Answer: $\left(\frac{635}{217}, \frac{1275}{496}\right)$ or $(2.93, 2.57)$.

Question 10

Most candidates knew that to obtain the reduction formula they needed to integrate by parts and this was done with varying degrees of success but often correctly. Knowledge of mean values was less secure and obtaining an integral, in terms of *t*, was only achieved by the best candidates. Even if this part was not completed, candidates were able to use the printed result, along with the reduction formula from the first part, in order to obtain the final value. This last part was done well by many candidates.

Answer: $\frac{3\pi}{32}a$.

Question 11 EITHER

At the beginning of this question, a significant number of those who attempted this alternative did not clearly state that $(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos 3\theta + \mathrm{i}\sin 3\theta$. They usually redeemed themselves by stating $\mathrm{Re}(\cos\theta + \mathrm{i}\sin\theta)^3 = \cos 3\theta$ and $\mathrm{Im}(\cos\theta + \mathrm{i}\sin\theta)^3 = \sin 3\theta$, either explicitly or implicitly. The derivation of $\tan 3\theta$ was then usually sound, with only a few showing insufficient working for a displayed result. Some of the stated roots for $\tan 3\theta = 1$ were incorrect, or outside the required range. The majority of candidates realised that by setting $\tan 3\theta = 1$ the required cubic equation could be obtained. Solutions of the cubic equation from calculator functions was not acceptable and a significant proportion of candidates did not show sufficient detail as to how the exact roots, and hence how the exact trigonometric ratios were obtained.

Answers: $\frac{1}{12}\pi, \frac{5}{12}\pi, \frac{3}{4}\pi$; $\tan\frac{1}{12}\pi$, $\tan\frac{5}{12}\pi$, $\tan\frac{3}{4}\pi$; $\tan\frac{1}{12}\pi = 2-\sqrt{3}$, $\tan\frac{5}{12}\pi = 2+\sqrt{3}$, $\tan\frac{3}{4}\pi = -1$.

Question 11 OR

This question was done well by those attempting this alternative. In part **(i)** the majority of candidates found the first derivative correctly. They then knew the technique of using the discriminant, from the resulting quadratic equation, when the derivative was equated to zero. The displayed result probably helped candidates to detect any errors that might have been made, since most solutions were completely correct. The majority of candidates were able to find the asymptotes correctly in part **(ii).** In both sketch graphs, common errors were not placing the oblique asymptote correctly for the appropriate value of λ, and for branches of the curves to be veering away from the asymptotes to an appreciable extent. In part **(iii)** some graphs had upper branches which did not cross the *x*-axis

Answers: **(i)** $\dfrac{x^2+6x+3\lambda+6\lambda^2}{(x+3)^2}$; **(ii)** $x=-3$, $y = x+\lambda-3$.

FURTHER MATHEMATICS

Paper 9231/12
Paper 12

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Answers: **(i)** $\dfrac{x^2 + 6x + 3\lambda + 6\lambda^2}{(x+3)^2}$; **(ii)** $x = -3$, $y = x + \lambda - 3$.

FURTHER MATHEMATICS

Paper 9231/13

Paper 13

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Candidates displayed a sound knowledge of most topics on the syllabus. As well as the vector work, already mentioned, candidates tackled the questions on matrices, differential equations, roots of equations, differentiating implicit functions and graphs of rational functions confidently. The question on proof by induction was a cause of difficulty for some candidates.

Comments on specific questions

Question 1

The first part of the question was tackled competently, with only relatively few candidates having the wrong coefficient before the sum of the first n squares. There was a number of different approaches for the second part, with rather more cases of incorrect logic than in the first part.

Answers: $\dfrac{2n(n+1)(2n+1)}{3}$; $-n(2n+1)$.

Question 2

There were a good number of correct solutions. Some attempts were marred by not showing sufficient working, while others fell down by not stating an inductive hypothesis, or not showing that it was true in the case k = 1, or not writing a satisfactory conclusion. A small number of candidates had trouble multiplying matrices to prove the inductive step, and occasionally adding, rather than multiplying, was observed.

Question 3

This question was done very well by the vast majority of candidates. Nearly all obtained the correct cubic equation. The method of solving it was not always evident, leaving open the possibility that the roots had been obtained from a calculator.

Answers: $x^3+6x^2-x-30=0$; 2, –3, –5.

Question 4

The given implicit equation was accurately differentiated and the value of $\frac{dy}{dx}$ was duly obtained as –4 at the point (–1, 1), in most cases. Some candidates made $\frac{dy}{dx}$ the subject of their equation before differentiating a second time, but most continued to differentiate implicitly, with considerable accuracy. The occasional algebraic or arithmetic slip meant that fewer candidates obtained the correct value for the second derivative.

Answer: –42.

Question 5

A considerable number of candidates were unable to derive the reduction formula in the first part of the question, despite the clue in the wording of the question. The main problem seemed to be in the integration of $\tan^{n-2} x \sec^2 x$, where the power of a function and its derivative were not noted. A number avoided this difficulty by using a suitable substitution. In the second part, the majority of candidates were able to use the reduction formula together with I_0, or I_2 in order to obtain the printed result.

Question 6

Many of the polar graphs were satisfactory, although a minority of candidates drew $x = a$ instead of $r = a$. Those with correct graphs were invariably able to identify the value of β. A surprising number of candidates used integration to find one twelfth of the area of a circle. The integration in order to find the area of the sector which was to be added to $\frac{1}{12}\pi\alpha^2$ was mostly done well by use of the double angle formula, although there were some slips with signs or inserting limits, and a few candidates subtracted the two areas, rather than adding them.

Answer: $\frac{1}{6}\pi$.

Question 7

The first part of this question was mostly done well, with only relatively few errors in obtaining $\frac{ds}{dt}$. Most candidates were able to obtain a correct integral representation for the surface area, but only the best candidates could handle the repeated integration by parts, which enabled them to obtain the final answer. A number of candidates avoided the integration by parts by integrating $e^{(2+i)t}$ and taking the imaginary part. Decimal answers were acceptable, if obtained legitimately and not from a calculator function.

Answers: $\sqrt{2}(e^{\pi} - 1)$; $\frac{2\sqrt{2}}{5}(e^{2\pi} + 1)$.

Question 8

This question was done well by many candidates. Most were able to find the complementary function correctly and many the particular integral, with only a few making errors with coefficients for the particular integral. One of the arbitrary constants was easily found by most candidates, but there some errors when the initial conditions had to be substituted into the derivative in order to obtain the second arbitrary constant. Correct logic enabled the final mark to be obtained, even if errors had been made in determining the arbitrary constants.

Answers: $x = e^{-t}(A\cos 2t + B\sin 2t) + 2\sin t - \cos t$; $x = e^{-t}(6\cos 2t + 3\sin 2t) + 2\sin t - \cos t$;
$x \approx 2\sin t - \cos t$

Question 9

This question was generally well done. Most candidates answered parts **(i)** and **(ii)** correctly, in some cases in spite of slips in the polynomial division. More errors appeared in part **(iii)** however, either because candidates did not apply the quotient rule correctly to the displayed expression for y, or following a slip in the division in part **(ii)**. The majority of those who answered part **(iii)** correctly went on to obtain the correct result in part **(iv)**.

Answers: **(i)** $x = 1$; **(ii)** 2; **(iii)** 7; **(iv)** $-7 < k < 1$.

Question 10

Most candidates correctly obtained the equation of the plane in the first part of this question. The majority then correctly wrote $\mathbf{r} = \mathbf{i} + 10\mathbf{j} + 3\mathbf{k} + t(5\mathbf{i} - 3\mathbf{j} - 2\mathbf{k})$ as the equation of the perpendicular and then used $\mathbf{r}.(5\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = 7$ to obtain $t = 1$ and the position vector of the foot of the perpendicular as required. A sizeable minority luckily guessed $t = 1$ here without writing the general equation of the perpendicular. These candidates lost marks, having failed to justify their result. Many candidates obtained the shortest distance between l_1 and l_3 correctly and hence scored all the marks on the last part of the question

Answers: $5x - 3y - 2z = 7$; $6\mathbf{i} + 7\mathbf{j} + \mathbf{k}$; $\dfrac{10}{\sqrt{42}}$.

Question 11 EITHER

This was marginally more popular than the other alternative and attracted two different approaches to the first part. Some candidates correctly identified matrices **P** and **D** and wrote $\mathbf{A} = \mathbf{PDP}^{-1}$. Mistakes were evident in the evaluation of $\mathbf{P}^{-1}$ however, and only the best candidates got the matrix **A** fully correct. Others solved three sets of linear equations arising from $\mathbf{Ae} = \lambda\mathbf{e}$ in order to obtain **A**, and then realised they needed matrices **P** and **D** for the second part. Here most candidates correctly wrote $\mathbf{A}^{2n} = \mathbf{P}\,\mathbf{D}^{2n}\,\mathbf{P}^{-1}$ and, of those who correctly obtained **A** in the first part, a high proportion went on to obtain a correct form for $\mathbf{A}^{2n}$.

Answers: **(i)** $\begin{pmatrix} 1.5 & 0.5 & 0.5 \\ 1.5 & 0.5 & 1.5 \\ -1 & 1 & 0 \end{pmatrix}$; **(ii)** $\dfrac{1}{2}\begin{pmatrix} 2^{2n}+1 & 2^{2n}-1 & 2^{2n}-1 \\ 2^{2n}-1 & 2^{2n}+1 & 2^{2n}-1 \\ 0 & 0 & 2 \end{pmatrix}$.

Question 11 OR

Those attempting this alternative usually managed to reduce the matrix to echelon form and obtain its rank correctly. A small number of candidates stopped short of echelon form. Only the best candidates were able to produce a rigorous proof for the second part. Some produced a reverse argument, which constituted a verification, rather than a proof, which earned 2 of the 4 marks. A good number were able to find the values of p, q and r correctly in the next part of the question and go on to use them in the final part to find **x** correctly. A small number did not realise the need to use the values from the third part in the final part.

Answers: 3; 1, –1, –1; $\mathbf{x} = \begin{pmatrix} 1.25 \\ -0.75 \\ -0.75 \\ 0.25 \end{pmatrix}$..

FURTHER MATHEMATICS

Paper 9231/21

Paper 21

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General comments

Most candidates attempted all the questions, and while some very good answers were seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 11**, there was a general preference for the Statistics alternative, though the minority of candidates who chose the Mechanics alternative often produced good attempts. Indeed all questions were answered well by some candidates, most frequently **Questions 1**, **4**, **6** and **7**.

Candidates are encouraged to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable. In some of the Mechanics questions it is helpful to include diagrams which show, for example, where forces are acting, and also the directions of forces as in **Question 3** and the directions of motion of bodies, as in **Question 4**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. Thus in **Question 4** a variety of speeds may be considered, with a potential for confusion if they are not clearly defined, while in **Questions 7**, **9** and **10** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need, referred to above, to explain their working clearly in those questions which require certain given results to be shown to be true rather than finding unknown results. In these circumstances it is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, such as a numerical value in a Statistics question, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

Comments on specific questions

Question 1

Candidates almost always appreciated that the key to this question on simple harmonic motion is to make use of the standard formula $v^2 = \omega^2(a^2 - x^2)$, since substitution of the given distances and speeds in place of x and v gives two simultaneous equations for ω and the amplitude a, the values of which may be found in either order. The period T then follows from another SHM equation $T = \frac{2\pi}{\omega}$, also widely known to the candidates. Apart from arithmetical errors, the only apparent obstacle to obtaining full marks for this question was a relatively rare misquoting of the initial SHM equation.

Answers: $\frac{1}{2}\pi$ s; 5 m.

Question 2

An exact expression for the tension T comes from an application of Hooke's law, and includes the half-length $\sqrt{(a^2 + x^2)}$ of the extended string. This approximates to a when x^2 is neglected, yielding the required expression for T. Since this expression is given in the question, it is not sufficient to derive it by effectively taking the extended half-length to be a without any attempt at an explanation. The second part also presented problems, with the frequent introduction of a force mg even though the string is on a horizontal surface, or a consideration of motion along AB rather than perpendicular to it. The correct approach is to apply Newton's law of motion, with the relevant force here equal to $\frac{2Tx}{\sqrt{a^2+x^2}}$, which becomes simply $\frac{2Tx}{a}$ when x^2 is again neglected. Substitution for T using the earlier result produces the usual form of the SHM equation, from which the period may be found.

Answer: $2\pi\sqrt{\frac{alm}{2\lambda(a-l)}}$.

Question 3

Making use of the given value $\frac{1}{2}$ of the coefficient of friction suggested to most candidates finding the ratio of the friction F and normal reaction R at the point of contact between the plane and the hemisphere. Although this can be achieved by resolving forces along and perpendicular to the plane to find F and R respectively, some candidates sensibly resolved horizontally instead to give $F \cos \alpha = R \sin \alpha$ and hence $\frac{F}{R} = \tan \alpha$. The required inequality for $\tan \alpha$ then follows from $\frac{F}{R} < \frac{1}{2}$. The second part requires a relation between m and M, which can be found by taking moments about the point of contact, or more easily about the centre of the hemisphere provided F has previously been found. The required inequality is then most easily derived at the expense of some algebraic manipulation by noting that if $\tan \alpha < \frac{1}{2}$ then $\sin \alpha < \frac{1}{\sqrt{5}}$. Those candidates who are unhappy with the manipulation of such inequalities and instead use $\tan \alpha = \frac{1}{2}$ in order to produce the limiting value of $\frac{m}{M}$, need to then justify the given inequality.

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Almost all candidates rightly appreciated that both parts of this question require the application of conservation of momentum and Newton's restitution equation, together with appropriate substitution for the masses and speeds of the spheres when colliding together. They were usually also aware that B's rebounding from the wall reverses its direction and alters its speed by a factor e. As a result the question was generally well answered, with such errors as occurred being usually of detail rather than lack of understanding.

Answers: $\frac{3}{5}u$, $\frac{6}{5}u$; $\frac{3}{4}$.

Question 5

The key to analysing what occurs when the particle P loses contact with the sphere is to realise that the sphere will cease to exert any force on P and so the radial force $mg\cos\theta$ will be equal to $\frac{mv^2}{a}$ at that moment. Substitution for the speed v at this point using conservation of energy relative to the initial position yields the given value for $\cos\theta$ and also the required value of v. An equally valid approach is to show that when $\cos\theta$ takes this given value the force exerted by the sphere on the particle is zero. The second part, concerning the motion after P leaves the surface of the sphere, requires more thoughtful analysis. One approach is to first consider the horizontal component of the motion of P in order to find the time to move from $\frac{4a}{5}$ to $\frac{7a}{5}$ from the vertical diameter, and then show that in the same time it will fall from $\frac{3a}{5}$ above the vertical diameter to $\frac{31a}{30}$ below it. Alternatively and equivalently, the standard form of the trajectory equation can be used, and both approaches were successfully applied by candidates.

Answer: $\sqrt{\frac{3ag}{5}}$ or $\sqrt{6a}$.

Question 6

Most candidates made a good attempt at this question, finding the expected values correctly and then comparing the calculated value 5.36 of χ^2 with the tabular value 5.991 in order to conclude that the null hypothesis should be accepted. As in all similar tests it is necessary to formulate appropriate hypotheses, and in this case the null hypothesis is of no association between the area in which residents live and their preference as to how the money should be spent.

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The null and alternative hypotheses were usually stated correctly, though candidates must express these in terms of the population and not the sample mean, for example by simply writing $\mu = 1.2$ and $\mu > 1.2$. Similarly the stated assumption of normality needs to refer unambiguously to the population of masses. Most candidates went on to find the sample mean and to estimate the variance of the population, and hence calculate a t-value of 1.08. Comparison with the critical t-value of 1.383 then leads to the conclusion that the greengrocer's claim is not supported.

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Estimating the population variance and finding the confidence interval presented no problems to most candidates, apart from occasional confusion over whether to use a biased or an unbiased estimate in the confidence interval formula employed, or over the appropriate tabular t-value, here 2.571. The second part was rather more challenging, though candidates need only formulate the inequality $\frac{1.96 \times 5.6}{\sqrt{n}} < 2.5$ and then find the smallest integer satisfying it. Those who choose to use an equality instead, for which the solution is 19.3, need to consider whether the required answer is 19 or 20. Among the possible errors to be avoided in the second part of the question are taking the interval half-width to be 5 rather than 2.5, using a critical value other than 1.96, mistaking 5.6 as being the variance rather than the standard deviation, and employing an estimated standard deviation from the first part of the question in place of 5.6.

Answers: [29·4, 40·9]; 20.

Question 9

The null and alternative hypotheses should as usual be stated in terms of the relationship between the means of the two populations, here the times taken by boys and by girls to complete the puzzle. If boys are slower than girls, then the mean time taken by boys is of course greater than for girls, and not the reverse. After using the given sample data to estimate the variances of the two populations, candidates should use these two values to produce a combined estimate 0.00541, from $\frac{s_b{}^2}{40}+\frac{s_g{}^2}{60}$ for example. Dividing the difference of the two sample means by the estimated standard deviation then yields a *z*-value of 1.86. An often seen alternative, though with the necessary assumption of equal variances not always stated, was to use a pooled estimate 0.132 of the common variance, leading to a *z*-value of 1.85. In either case comparison with the tabular *z*-value 1.96 leads to rejection of the alternative hypothesis and thus the conclusion that Mr Lee's assertion is not correct.

Question 10

Calculation of the product moment correlation coefficient r for the data and the equation of a regression line are both straightforward, with relevant expressions given in the *List of Formulae*. Apart from an occasional arithmetical error most candidates experienced little difficulty. The test requires an explicit statement of the null and alternative hypotheses, $\rho = 0$ and $\rho \neq 0$, and here candidates should be aware that r and ρ are not the same entity. Comparison of the magnitude of the previously calculated value of r with the tabular value 0.708 leads to a conclusion of there being evidence of non-zero correlation. There is no strong reason for preferring either the regression line of x on y or that of y on x, but in either case it is the value of y when $x = 2$ which is required and not x when $y = 2$.

Answers: **(i)** 0.750; **(iii)** 3.11 or 3.28.

Question 11 EITHER

Although it was less popular than the Statistics alternative discussed below, some candidates made very good attempts at this optional question. The given moment of inertia of the system may be verified by first finding the moment of inertia of a single side of the square about O using the parallel axes theorem, and then adding the equal moments of inertia of all four sides to that of the ring about its centre O. Candidates should not of course attempt to use the standard formula for the moment of inertia of a square lamina. Most candidates realised that the perpendicular axes theorem should be employed when finding the moment of inertia about the axis l through A since l is in the plane of the body rather than perpendicular to it, but the theorem needs to be applied to axes at O rather than A, and then the parallel axes theorem used to transfer from O to A. Application of the two theorems in the opposite order is invalid, giving an incorrect result $\frac{17Ma^2}{6}$ which was not uncommon. The final part was somewhat more challenging. While most candidates probably realised that P attains its greatest speed when vertically below the axis, not stating this fact or indicating it on a diagram risks losing credit for their realisation if their subsequent working is both incorrect and unclear. This working should be based on equating the rotational energy acquired by the system to the potential energy lost, with the latter found either by considering the respective distances fallen by the three individual centres of mass or that fallen by the centre of mass of the system. Finding the greatest angular speed of the system in this way is not of course sufficient, since the speed of P required by the question is its linear speed, found by multiplying the system's angular speed by the distance $2a$ of P from the axis of rotation.

Answers: $\frac{13Ma^2}{3}$; $6\sqrt{\frac{ga}{5}}$.

Question 11 OR

The value of k follows from the requirement that the area under the graph must equal 1. Knowing then that the common point of the two straight line segments is $(1, \frac{2}{3})$ enables the equation of each of these lines to be written down, effectively giving the probability density function f(x) which may be integrated to verify the given form of F(x). When finding F(x) for $1 < x < 3$ by integrating $f(x) = 1 - \frac{1}{3}x$ between 1 and x, candidates need to remember to add $F(1) = \frac{1}{3}$ since otherwise the constant term in their result is incorrect. If candidates cannot find the equations of the lines on the graph of f(x) from the information given about the graph, it is insufficient to deduce them by differentiating F(x), since the objective is to verify F(x). The distribution function G(y) of Y is found from $F(y^{\frac{1}{2}})$, and differentiation then gives the required probability density function g(y) of Y. Equating G(y) to $\frac{1}{2}$ and solving the resulting quadratic gives the median value of Y, which must lie in the interval (1, 9).

Answers: $\frac{2}{3}$; **(i)** $\frac{1}{3}$ $(0 < y < 1)$, $\frac{1}{2}y^{-\frac{1}{2}} - \frac{1}{6}$ $(1 < y < 9)$; **(ii)** 1.61.

FURTHER MATHEMATICS

Paper 9231/22

Paper 22

Key messages

In order that candidates receive credit where possible for incorrect answers, they should remember to show full working and reasoning. Incorrect answers without working cannot be given any credit, whereas partial credit can be awarded if a correct method is shown. Where candidates are asked to 'show' or 'prove' a result, marks will be lost if any essential steps in the argument are omitted.

General comments

Most candidates attempted all the questions, and while some very good answers were seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 11**, there was a general preference for the Statistics alternative, though the minority of candidates who chose the Mechanics alternative often produced good attempts. Indeed all questions were answered well by some candidates, most frequently **Questions 1**, **4**, **6** and **7**.

Candidates are encouraged to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable. In some of the Mechanics questions it is helpful to include diagrams which show, for example, where forces are acting, and also the directions of forces as in **Question 3** and the directions of motion of bodies, as in **Question 4**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. Thus in **Question 4** a variety of speeds may be considered, with a potential for confusion if they are not clearly defined, while in **Questions 7**, **9** and **10** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need, referred to above, to explain their working clearly in those questions which require certain given results to be shown to be true rather than finding unknown results. In these circumstances it is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, such as a numerical value in a Statistics question, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

Comments on specific questions

Question 1

Candidates almost always appreciated that the key to this question on simple harmonic motion is to make use of the standard formula $v^2 = \omega^2(a^2 - x^2)$, since substitution of the given distances and speeds in place of x and v gives two simultaneous equations for ω and the amplitude a, the values of which may be found in either order. The period T then follows from another SHM equation $T = \frac{2\pi}{\omega}$, also widely known to the candidates. Apart from arithmetical errors, the only apparent obstacle to obtaining full marks for this question was a relatively rare misquoting of the initial SHM equation.

Answers: $\frac{1}{2}\pi$ s; 5 m.

Question 2

An exact expression for the tension T comes from an application of Hooke's law, and includes the half-length $\sqrt{(a^2 + x^2)}$ of the extended string. This approximates to a when x^2 is neglected, yielding the required expression for T. Since this expression is given in the question, it is not sufficient to derive it by effectively taking the extended half-length to be a without any attempt at an explanation. The second part also presented problems, with the frequent introduction of a force mg even though the string is on a horizontal surface, or a consideration of motion along AB rather than perpendicular to it. The correct approach is to apply Newton's law of motion, with the relevant force here equal to $\frac{2Tx}{\sqrt{a^2+x^2}}$, which becomes simply $\frac{2Tx}{a}$ when x^2 is again neglected. Substitution for T using the earlier result produces the usual form of the SHM equation, from which the period may be found.

Answer: $2\pi\sqrt{\frac{alm}{2\lambda(a-l)}}$.

Question 3

Making use of the given value $\frac{1}{2}$ of the coefficient of friction suggested to most candidates finding the ratio of the friction F and normal reaction R at the point of contact between the plane and the hemisphere. Although this can be achieved by resolving forces along and perpendicular to the plane to find F and R respectively, some candidates sensibly resolved horizontally instead to give $F \cos \alpha = R \sin \alpha$ and hence $\frac{F}{R} = \tan \alpha$. The required inequality for $\tan \alpha$ then follows from $\frac{F}{R} < \frac{1}{2}$. The second part requires a relation between m and M, which can be found by taking moments about the point of contact, or more easily about the centre of the hemisphere provided F has previously been found. The required inequality is then most easily derived at the expense of some algebraic manipulation by noting that if $\tan \alpha < \frac{1}{2}$ then $\sin \alpha < \frac{1}{\sqrt{5}}$.

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The value of k follows from the requirement that the area under the graph must equal 1. Knowing then that the common point of the two straight line segments is $(1, \frac{2}{3})$ enables the equation of each of these lines to be written down, effectively giving the probability density function f(x) which may be integrated to verify the given form of F(x). When finding F(x) for $1 < x < 3$ by integrating $f(x) = 1 - \frac{1}{3}x$ between 1 and x, candidates need to remember to add $F(1) = \frac{1}{3}$ since otherwise the constant term in their result is incorrect. If candidates cannot find the equations of the lines on the graph of f(x) from the information given about the graph, it is insufficient to deduce them by differentiating F(x), since the objective is to verify F(x). The distribution function G(y) of Y is found from $F(y^{\frac{1}{2}})$, and differentiation then gives the required probability density function g(y) of Y. Equating G(y) to $\frac{1}{2}$ and solving the resulting quadratic gives the median value of Y, which must lie in the interval (1, 9).

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FURTHER MATHEMATICS

Paper 9231/23

Paper 23

Key messages

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Most candidates attempted all the questions, and while some very good answers were seen, the paper discriminated well between different levels of ability. In the only question which offered a choice, namely **Question 10**, there was a strong preference for the Statistics alternative, though the small minority of candidates who chose the Mechanics alternative frequently produced good attempts. Indeed all questions were answered well by some candidates, most frequently **Questions 5, 6** and **9**. The parts of those questions which involved inequalities were widely found to be challenging, as in **Questions 1, 4** and **7**.

Candidates are encouraged to set out their work clearly, with any corrections legible and the replacements to deleted attempts readily identifiable. In some of the Mechanics questions it is helpful to include diagrams which show, for example, where forces are acting, and also the directions of forces as in **Question 3** and the directions of motion of particles, as in **Question 1**. Where the meaning of symbols introduced by candidates is not clearly defined in this way or is not otherwise obvious, it is helpful to include a definition or explanation. Thus in **Questions 1** and **4** a variety of speeds may be considered, with a potential for confusion if they are not defined, while in **Questions 8** and **9** any symbols used when stating the hypotheses should either be in standard notation or should be correctly defined.

Many candidates appreciated the need, referred to above, to explain their working clearly in those questions which require certain given results to be shown to be true rather than finding unknown results. In these circumstances it is particularly advisable to explain or justify any new equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, such as a numerical value in a Statistics question, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

Comments on specific questions

Question 1

Almost all candidates rightly appreciated that both parts of this question require the application of conservation of momentum and Newton's restitution equation, together with appropriate substitution for some of the speeds of the particles involved in the two collisions. Thus in the first collision, between *A* and *B*, the speeds of *B* beforehand and *A* afterwards are both zero, so that the given value of *k* may be readily deduced by combining the two simplified equations. *C* is also stationary before the subsequent collision with *B*, and if candidates choose to explicitly or implicitly assume that *B* is brought to rest then they may easily obtain the limiting value of $\frac{1}{3}$ for *e* but they should be aware that this is not sufficient to show that $e < \frac{1}{3}$. Uncertainty over the handling of inequalities was also evident in other questions, and candidates should appreciate that if they choose to find only a limiting value, then they should also determine whether it is an upper or lower limit. Instead it is probably advisable here to take the speed of *B* after the second collision as non-zero, then use the requirement that *B* does not change direction in order to produce the required inequality for *e*. This can be done in a variety of ways, one of which is to combine the two equations in order to find the speed of *B* after the collision in terms of *e*, and then consider the sign of the factor $1 - 3e$ in it.

Question 2

The required moment of inertia of the system was found correctly by most candidates, who applied the parallel axes theorem to each of the two discs. The second part was more challenging, however. While most candidates probably realised that P attains its greatest speed when vertically below the axis, not stating this fact or indicating it on a diagram risks losing credit for their realisation if their subsequent working is both incorrect and unclear. This working should be based on equating the rotational energy acquired by the system to the potential energy lost, with the latter found either by considering the respective distances fallen by the three individual centres of mass or that fallen by the centre of mass of the system. Finding the greatest angular speed in this way is not of course sufficient, since the speed of P required by the question is its linear speed, found by multiplying its angular speed by the distance $8a$ of P from the axis of rotation.

Answer: 9.9 ms^{-1}.

Question 3

As in similar questions, the amount of work required can be minimised by first giving some thought to which of the several possible moment equations or resolution of forces will yield the required results most directly. Here the normal contact force at B is found most easily by taking moments for the system about C, and then that at C by a vertical resolution of forces for the system, though other methods can be used. The natural next step is to find the frictional force at B or C, perhaps by taking moments for the appropriate rod about A, and while the other frictional force can be found similarly, this is strictly unnecessary since a horizontal resolution of forces shows that the frictional forces at B and C must be equal and opposite. The limiting values of the coefficient of friction at B and C are not equal, though some candidates mistakenly assumed them to be so, and are most obviously found by taking the ratios of the respective friction and reaction. It is, incidentally, possible to determine the two limiting values without finding the frictional forces explicitly. The least possible value of μ is then the larger of the two limiting values, though it is tempting but incorrect to choose the smaller. A few candidates realised that since the frictional forces are equal at B and C, it is in fact only necessary to consider the point at which the normal contact force is smaller, namely B, in order to find the least possible value of μ.

Answer: $\frac{9}{13}$.

Question 4

Derivation of the given expression for the tension T requires that the net radial force acting on the particle as a result of T and $mg \cos \theta$ be equated to $\frac{mv^2}{a}$, followed by substitution for the speed v using conservation of energy relative to the initial position. The second part, concerning the motion after the string comes into contact with the peg, requires more thoughtful analysis, not least in formulating the condition for P to complete a vertical circle. Many candidates assumed that this merely requires P to have sufficient energy to reach the required height, and therefore found an inequality for x for which $w > 0$, or simply the limiting value for which $w = 0$, where w is the speed of the particle at a height $a - x$ above O, found by conservation of energy. This is insufficient, however, since the critical condition is that $T > 0$ at the highest point, and this requires a very similar resolution of radial forces to that in the first part of the question, with one force being simply mg and the other $\frac{mw^2}{a-x}$). Substitution for w leads to $x > \frac{2a}{3}$. Perhaps through being uncomfortable with inequalities, some candidates found instead the limiting value of x for which $T = 0$. This must of course be supplemented by some convincing argument as to why this is the least possible value. If, as was sometimes the case, the speed of the particle when the string first makes contact with the peg is also introduced, it is advisable to say what the various symbols for speeds represent, rather than simply writing down several equations without clear explanation.

Answer: $\frac{2a}{3}$.

Question 5

Some candidates recognized that f(x) is an exponential distribution with parameter $\lambda = 0.01$ and were thus able to immediately state the value of E(X) as $\frac{1}{\lambda}$, while others integrated xf(x) to arrive at the same result. The median value m was usually found correctly, by either stating or finding by integration the cumulative distribution function $F(x) = 1 - e^{-\lambda x}$ and then equating it to $\frac{1}{2}$ and solving for $x = m$. The final probability is found from $F(\frac{1}{\lambda}) - F(m)$, and is thus $\frac{1}{2} - \frac{1}{e}$.

Answers: **(i)** 100; **(ii)** 100 ln 2 or 69.3; **(iii)** 0.132.

Question 6

This was often answered correctly, if not always in the most direct manner. All that is required is to express the pooled estimate of the common variance in terms of the given information, equate this expression to 3, and choose the integral root of the resulting quadratic. It is unnecessary, if valid, to first find separate expressions for the estimates of the variances of the two distributions before combining them. There is then a risk of confusing biased and unbiased estimates when doing so, thereby introducing an invalid and complicating factor $n - 1$.

Answer: 10.

Question 7

Denoting the probability of a 6 appearing on one throw by p, and not appearing by q, the first two required probabilities are pq^4 and $1 - q^7$, as many candidates appreciated. The final part is a little less straightforward since it depends on formulating and solving the inequality $1 - q^{n-1} > 0.99$, and the apparent aversion of some candidates to inequalities meant that they solved instead the corresponding equality. If this is done, then they need to also justify their final value as being the least integer. Where errors occurred, they were usually the use of a wrong power, such as 6 or 8 in the second part or n in the final one. Although it may be tempting to approximate the results in the first two parts to 0.08 and 0.72, this should be resisted since the rubric for the paper specifies that such answers be given to 3 significant figures.

Answers: **(i)** 0.0804; **(ii)** 0.721; **(iii)** 27.

Question 8

The key to answering this question correctly is the selection of a suitable statistic. In this case a paired sample t-distribution is appropriate, so candidates should base their working on the differences between the two weights for each of the eight sampled employees, under the assumption stated in the question that the population of the differences in weight is normally distributed. The majority of candidates did indeed do so, and while some misused a biased estimate of the population variance in both parts, most calculated the sample mean 3.225 and the unbiased population variance 2.419 correctly, and used them with tabular t-values of 2.365 and 1.895 respectively for the confidence interval and the test. In the latter case, the calculated t-value of 1.32 leads to the conclusion that there is not a reduction of more than 2.5 kg in the population mean weight. In framing the hypotheses for this test, it is important to make clear by an appropriate use of symbols, such as the conventional μ, that these are based on the mean weight of the population and not some measure of the sample.

Answer: [1.92, 4.53].

Question 9

Calculation of the mean values, the equation of a regression line and the product moment correlation coefficient r for the data are all straightforward, with relevant expressions given in the *List of Formulae*. Apart from an occasional arithmetical error most candidates experienced little difficulty. There is no strong reason for preferring either the regression line of x on y or that of y on x, but in either case it is the value of x when $y = 56$ which is required and not y when $x = 56$ as some mistakenly found. The final test requires an explicit statement of the null and alternative hypotheses, $\rho = 0$ and $\rho \neq 0$, and here candidates should be aware that r and ρ are not the same entity. Finally comparison of the magnitude of the previously calculated value of r with the tabular value 0.514 leads to a conclusion of there being evidence of a non-zero correlation coefficient.

Answers: **(i)** 50.1, 51.5; **(ii)** 51 or 53; **(iii)** 0.598.

Question 10 EITHER

Although much less popular than the Statistics alternative discussed below, some candidates made very good attempts at this optional question. Applying Newton's law when the string is at a general distance x from the equilibrium position shows that the acceleration of the particle is equal to a constant multiple $-\frac{\lambda}{ma}$ of x, where the string has modulus of elasticity λ and natural length a. Thus the particle performs simple harmonic motion, enabling the period to be derived using the standard SHM formula $T = \frac{2\pi}{\omega}$ with here $\omega^2 = \frac{\lambda}{ma}$. Equating the two vertical forces on the particle at the equilibrium position shows that $\frac{\lambda}{ma} = \frac{g}{d}$, and substitution yields T in the given form. Simply asserting that $\omega^2 = \frac{g}{d}$ in order to find T without demonstrating that the motion is simple harmonic is not a sufficient proof. The required time in the second part of the question is found by adding together the time from rest to the point at which the string becomes slack and the subsequent time to the first instantaneous rest. The first of these follows from solving $x = 2d \cos \omega t$ with $x = -d$, though some candidates used instead $x = 2d \sin \omega t$ with $x = d$ and then added $\frac{1}{4}T$. The particle then moves under gravity with initial speed $v = \omega d\sqrt{3}$, so the time to instantaneous rest is simply $\frac{v}{g}$, and hence the given result.

Question 10 OR

Although this alternative was chosen by the majority of candidates and many were able to find the mean of the given sample, they also need to calculate the variance and to do so correctly. This enables them to sensibly comment on whether the two values are approximately equal, as would be expected of a Poisson distribution. The use of the appropriate Poisson formula to verify the given value of p usually presented no difficulty, and q was also often found correctly from a similar Poisson formula or by noting that the total of the expected number of days should be 200. Since each expected value used in the χ^2-test must be at least 5, candidates need to combine the last three intervals, as many indeed did. The χ^2-test is then conducted in the usual way, and comparison of 5.54 with the tabular value 6.251 leads to acceptance of the null hypothesis, namely that the specified Poisson distribution is indeed a suitable model.

Answers: 1.31, 1.21; **(i)** 20.219.