UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education Advanced Level

FURTHER MATHEMATICS **9231/01**

Paper 1 **October/November 2007**

3 hours

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF10)

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READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **5** printed pages and **3** blank pages.

1 A curve is defined parametrically by

$$x = at^2, \quad y = at,$$

where a is a positive constant. The part of the curve joining the point where $t = 0$ to the point where $t = \sqrt{2}$ is rotated through one complete revolution about the x-axis. Show that the area of the surface obtained is $\frac{13}{3}\pi a^2$. [4]

2 Express

$$\frac{2n+3}{n(n+1)}$$

in partial fractions and hence use the method of differences to find

$$\sum_{n=1}^{N} \frac{2n+3}{n(n+1)} \left(\frac{1}{3}\right)^{n+1}$$

in terms of N. [4]

Deduce the value of

$$\sum_{n=1}^{\infty} \frac{2n+3}{n(n+1)} \left(\frac{1}{3}\right)^{n+1}.$$ [1]

3 Prove by induction that, for all $n \geqslant 1$,

$$\frac{d^n}{dx^n}\left(e^{x^2}\right) = P_n(x)e^{x^2},$$

where $P_n(x)$ is a polynomial in x of degree n with the coefficient of x^n equal to 2^n. [6]

4 The roots of the equation

$$x^3 - 8x^2 + 5 = 0$$

are α, β, γ. Show that

$$\alpha^2 = \frac{5}{\beta + \gamma}.$$ [4]

It is given that the roots are all real. Without reference to a graph, show that one of the roots is negative and the other two roots are positive. [3]

5 The positive variables x and y are related by

$$y = x^2 + 2\ln(xy).$$

Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when both x and y are equal to 1. [7]

6 The points A, B and C have position vectors $2\mathbf{i}$, $3\mathbf{j}$ and $4\mathbf{k}$ respectively. Find a vector which is perpendicular to the plane Π_1 containing A, B and C. [3]

The plane Π_2 has equation

$$\mathbf{r} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} - \mathbf{j}) + \mu(\mathbf{j} - \mathbf{k}).$$

Find the acute angle between the planes Π_1 and Π_2. [5]

7 The curve C has polar equation

$$r = \theta \sin \theta,$$

where $0 \leqslant \theta \leqslant \pi$. Draw a sketch of C. [2]

Find the area of the region enclosed by C, leaving your answer in terms of π. [7]

8 Let $I_n = \int_0^{\ln 2} (\mathrm{e}^x + \mathrm{e}^{-x})^n \, \mathrm{d}x$.

(i) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(\mathrm{e}^x - \mathrm{e}^{-x})(\mathrm{e}^x + \mathrm{e}^{-x})^{n-1}\right] = n(\mathrm{e}^x + \mathrm{e}^{-x})^n - 4(n-1)(\mathrm{e}^x + \mathrm{e}^{-x})^{n-2}.$$ [3]

(ii) Hence show that

$$nI_n = 4(n-1)I_{n-2} + \tfrac{3}{2}\left(\tfrac{5}{2}\right)^{n-1}.$$ [2]

(iii) Use the result in part **(ii)** to find the y-coordinate of the centroid of the region bounded by the axes, the line $x = \ln 2$ and the curve

$$y = (\mathrm{e}^x + \mathrm{e}^{-x})^2.$$

Give your answer correct to 3 decimal places. [5]

9 Write down, in any form, all the roots of the equation $z^5 - 1 = 0$. [2]

Hence find all the roots of the equation

$$(w-1)^4 + (w-1)^3 + (w-1)^2 + w = 0,$$

and deduce that none of them is real. [4]

Find the arguments of the two roots which have the smaller modulus. [4]

10 The vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$, $\mathbf{b}_4$ are defined as follows:

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{b}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{b}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

The linear space spanned by $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ is denoted by V_1 and the linear space spanned by $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_4$ is denoted by V_2.

(i) Give a reason why $V_1 \cup V_2$ is not a linear space. [1]

(ii) State the dimension of the linear space $V_1 \cap V_2$ and write down a basis. [2]

Consider now the set V_3 of all vectors of the form $q\mathbf{b}_2 + r\mathbf{b}_3 + s\mathbf{b}_4$, where q, r, s are real numbers. Show that V_3 is a linear space, and show also that it has dimension 3. [3]

Determine whether each of the vectors

$$\begin{pmatrix} 4 \\ 4 \\ 2 \\ 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5 \\ 4 \\ 2 \\ 5 \end{pmatrix}$$

belongs to V_3 and justify your conclusions. [4]

11 Find the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 4 \\ 1 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix}$$

and corresponding eigenvectors. [7]

The matrix $\mathbf{B}$ is defined by

$$\mathbf{B} = \mathbf{A} - k\mathbf{I},$$

where $\mathbf{I}$ is the 3×3 identity matrix and k is a real number. Find a non-singular matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{B}^3 = \mathbf{PDP}^{-1}.$$ [4]

12 Answer only **one** of the following two alternatives.

EITHER

The curve C has equation

$$y = \frac{ax^2 + bx + c}{x + 4},$$

where a, b and c are constants. It is given that $y = 2x - 5$ is an asymptote of C.

(i) Find the values of a and b. [3]

(ii) Given also that C has a turning point at $x = -1$, find the value of c. [3]

(iii) Find the set of values of y for which there are no points on C. [4]

(iv) Draw a sketch of the curve with equation

$$y = \frac{2(x-7)^2 + 3(x-7) - 2}{x - 3}.$$ [3]

[You should state the equations of the asymptotes and the coordinates of the turning points.]

OR

Show that the substitution $y = \dfrac{1}{w}$ reduces the differential equation

$$y\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^2 - 5y^2 = (5x^2 + 4x + 2)y^3$$

to

$$\frac{d^2w}{dx^2} + 2\frac{dw}{dx} + 5w = -5x^2 - 4x - 2.$$ [4]

Find the general solution for w in terms of x. [6]

Find a function f such that $\lim\limits_{x\to\infty}\left(\dfrac{y}{f(x)}\right) = 1$. [3]

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