UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education Advanced Level

FURTHER MATHEMATICS **9231/01**

Paper 1

October/November 2006

3 hours

Additional Materials: Answer Booklet/Paper
Graph paper
List of Formulae (MF10)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of **5** printed pages and **3** blank pages.

UNIVERSITY *of* CAMBRIDGE
International Examinations

1 It is given that

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{pmatrix}.$$

Write down the eigenvalues of **A** and find corresponding eigenvectors. [5]

2 The integral I_n, where n is a non-negative integer, is defined by

$$I_n = \int_0^1 x^n e^{-x^3}\, dx.$$

By considering $\frac{d}{dx}\left(x^{n+1}e^{-x^3}\right)$ or otherwise, show that

$$3I_{n+3} = (n+1)I_n - e^{-1}.$$ [3]

Hence find I_6 in terms of e and I_0. [2]

3 Verify that if

$$v_n = n(n+1)(n+2)\ \ldots\ (n+m),$$

then

$$v_{n+1} - v_n = (m+1)(n+1)(n+2)\ \ldots\ (n+m).$$ [2]

Given now that

$$u_n = (n+1)(n+2)\ \ldots\ (n+m),$$

find $\sum_{n=1}^{N} u_n$ in terms of m and N. [3]

4 Prove by mathematical induction that, for all positive integers n, $10^{3n} + 13^{n+1}$ is divisible by 7. [5]

5 Show that if $a \neq 3$ then the system of equations

$$\begin{aligned} 2x + 3y + 4z &= -5, \\ 4x + 5y - z &= 5a + 15, \\ 6x + 8y + az &= b - 2a + 21, \end{aligned}$$

has a unique solution. [3]

Given that $a = 3$, find the value of b for which the equations are consistent. [3]

6 The roots of the equation

$$x^3 + x + 1 = 0$$

are α, β, γ. Show that the equation whose roots are

$$\frac{4\alpha+1}{\alpha+1}, \quad \frac{4\beta+1}{\beta+1}, \quad \frac{4\gamma+1}{\gamma+1}$$

is of the form

$$y^3 + py + q = 0,$$

where the numbers p and q are to be determined. [5]

Hence find the value of

$$\left(\frac{4\alpha+1}{\alpha+1}\right)^n + \left(\frac{4\beta+1}{\beta+1}\right)^n + \left(\frac{4\gamma+1}{\gamma+1}\right)^n,$$

for $n = 2$ and for $n = 3$. [4]

7 The curve C has equation

$$r = 10\ln(1+\theta),$$

where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. Draw a sketch of C. [2]

Use the substitution $w = \ln(1+\theta)$ to show that the area of the sector bounded by the line $\theta = \frac{1}{2}\pi$ and the arc of C joining the origin to the point where $\theta = \frac{1}{2}\pi$ is

$$50(b^2 - 2b + 2)e^b - 100,$$

where $b = \ln(1 + \frac{1}{2}\pi)$. [6]

8 Given that

$$2y^3\frac{d^2y}{dx^2} + 12y^3\frac{dy}{dx} + 6y^2\left(\frac{dy}{dx}\right)^2 + 17y^4 = 13e^{-4x}$$

and that $v = y^4$, show that

$$\frac{d^2v}{dx^2} + 6\frac{dv}{dx} + 34v = 26e^{-4x}.$$ [4]

Hence find the general solution for y in terms of x. [5]

9 With O as origin, the points A, B, C have position vectors

$$\mathbf{i}, \quad \mathbf{i}+\mathbf{j}, \quad \mathbf{i}+\mathbf{j}+2\mathbf{k}$$

respectively. Find a vector equation of the common perpendicular of the lines AB and OC. [6]

Show that the shortest distance between the lines AB and OC is $\frac{2}{5}\sqrt{5}$. [2]

Find, in the form $ax + by + cz = d$, an equation for the plane containing AB and the common perpendicular of the lines AB and OC. [3]

10 The curve C has equation

$$y = x^2 + \lambda \sin(x + y),$$

where λ is a constant, and passes through the point $A\left(\frac{1}{4}\pi, \frac{1}{4}\pi\right)$. Show that C has no tangent which is parallel to the y-axis. [5]

Show that, at A,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 2 - \tfrac{1}{64}\pi(4 - \pi)(\pi + 2)^2. \qquad [5]$$

11 Prove de Moivre's theorem for a positive integral exponent:

$$\text{for all positive integers } n, \quad (\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta. \qquad [5]$$

Use de Moivre's theorem to show that

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta. \qquad [4]$$

Hence obtain the roots of the equation

$$128x^7 - 224x^5 + 112x^3 - 14x + 1 = 0$$

in the form $\cos q\pi$, where q is a rational number. [4]

12 Answer only **one** of the following two alternatives.

EITHER

The curve C has equation

$$y = \frac{x^2 + qx + 1}{2x + 3},$$

where q is a positive constant.

(i) Obtain the equations of the asymptotes of C. [3]

(ii) Find the value of q for which the x-axis is a tangent to C, and sketch C in this case. [4]

(iii) Sketch C for the case $q = 3$, giving the exact coordinates of the points of intersection of C with the x-axis. [3]

(iv) It is given that, for all values of the constant λ, the line

$$y = \lambda x + \tfrac{3}{2}\lambda + \tfrac{1}{2}(q - 3)$$

passes through the point of intersection of the asymptotes of C. Use this result, with the diagrams you have drawn, to show that if $\lambda < \tfrac{1}{2}$ then the equation

$$\frac{x^2 + qx + 1}{2x + 3} = \lambda x + \tfrac{3}{2}\lambda + \tfrac{1}{2}(q - 3)$$

has no real solution if q has the value found in part **(ii)**, but has 2 real distinct solutions if $q = 3$. [4]

OR

The curve C has equation

$$y = x^{\frac{1}{2}} - \tfrac{1}{3}x^{\frac{3}{2}} + \lambda,$$

where $\lambda > 0$ and $0 \leqslant x \leqslant 3$. The length of C is denoted by s. Prove that $s = 2\sqrt{3}$. [4]

The area of the surface generated when C is rotated through one revolution about the x-axis is denoted by S. Find S in terms of λ. [5]

The y-coordinate of the centroid of the region bounded by C, the axes and the line $x = 3$ is denoted by h. Given that $\int_0^3 y^2\,dx = \frac{3}{4} + \frac{8\sqrt{3}}{5}\lambda + 3\lambda^2$, show that

$$\lim_{\lambda \to \infty} \frac{S}{hs} = 4\pi. \qquad [5]$$

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